Isoperimetric Functions of Groups and their Subgroups

by © Richard Gaelan Hanlon

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0.1 Abstract

The objects of interest in this thesis are various isoperimetric functions of non–compact topological spaces. Such functions are of classical interest in Riemannian Geometry, and are used as group–theoretic invariants in Geometric Group Theory.

Very little is known about the general relationship between isoperimetric functions of a group and its subgroups, and it is this problem which this thesis aims to address. By using algebraic techniques, we will give sufficient conditions for an isoperimetric function of a subgroup $H \leq G$ to be bounded above by that of the ambient group. This result contrasts with known examples illustrating that this relationship does not always hold when our conditions are relaxed. We conclude our investigation by discussing applications to hyperbolic groups and how our main result can be used as a "negativity test" to show that a group does not contain particular subgroups.

0.2 Acknowledgements

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Chapter 1

Introduction

The objects we will be studying are various *isoperimetric functions* of (non–compact) topological spaces. Before we begin, we will mention some familiar examples of isoperimetric functions, describe some of their applications to Group Theory, and discuss our motivation for this thesis.

1.1 Familiar Isoperimetric Functions

If γ is a simple closed curve in the Euclidean plane $\mathbb{E}$ and A is the area of the region it encloses, then a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $A \leq f(L)$ for *all* simple closed curves γ of length $\leq L$ is called an *isoperimetric function* for $\mathbb{E}$. If γ has length L, then it is well known that

$$A \leq \frac{1}{4\pi} L^2$$

with equality holding only when γ is a circle. Thus the function $f(L) = \frac{1}{4\pi}L^2$ is an isoperimetric function for $\mathbb{E}$ and we say that $\mathbb{E}$ *satisfies a quadratic isoperimetric inequality.* If one has studied hyperbolic geometry, then he or she may recall that the geometry of the hyperbolic plane $\mathbb{H}$ is very different from that of $\mathbb{E}$ and that $\mathbb{H}$ satisfies a *linear* isoperimetric inequality.

If we consider a regular tiling of the Euclidean plane $\mathbb{E}$, then we can associate a group G which *acts* on $\mathbb{E}$ such that the tiling of $\mathbb{E}$ is preserved under the action of G. For example, consider the tiling of $\mathbb{E}$ by squares shown in Figure 1. Then the group $E = \langle\, x, y \mid xyx^{-1}y^{-1} = id\rangle$ acts on $\mathbb{E}$ by horizontal (x) and vertical (y) translations of unit length. Since this action of E on $\mathbb{E}$ is particularly nice (we call it a *geometric* action), we can use the quadratic isoperimetric inequality of $\mathbb{E}$ as an *invariant* of E and we say that the group E satisfies a quadratic isoperimetric inequality.

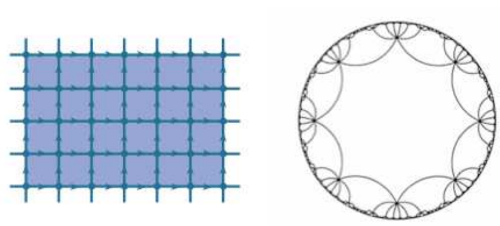


Figure 1.1: Regular tilings of the Euclidean (left) and Hyperbolic (right) planes. These pictures have been taken from the "Bridson's Universe of Finitely Presented Groups" blog entry at http://berstein.wordpress.com/

Similarly, we could consider a tiling of the hyperbolic plane $\mathbb{H}$ by regular octagons (see Figure 1.1). Here the group $H = \langle\ a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = id\ \rangle$ acts on $\mathbb{H}$ by unit translations in a similar sort of fashion to the action of E on $\mathbb{E}$. Once again, this action of H on $\mathbb{H}$ is particularly nice and we can use the linear isoperimetric function of $\mathbb{H}$ as an invariant of H. By using the fact that the isoperimetric function of H is strictly smaller than that of E, we can show that the group E *cannot* occur as a subgroup of H. This idea is portrayed in our main result Theorem 1.3.1.

1.2 Higher–Dimensional Isoperimetric Functions

The isoperimetric functions we previously mentioned measure the area required to fill a loop with a disk. However, we could also consider isoperimetric functions of higher dimensions which measure the volume required to fill 2–spheres with 3–balls, or more generally, n–manifolds with $(n+1)$–manifolds. Like the examples we just discussed, we can use these higher–dimensional isoperimetric functions of a space X as invariants of a group G, provided G acts "nicely" on X. This idea can be quite useful, and occasionally higher–dimensional isoperimetric functions yield information that lower–dimensional ones cannot.

For example, automatic groups are a particularly large class of groups that possess fast algorithms for computing topological and algebraic properties (see [14]). In particular, the isoperimetric functions of an automatic group are bounded above by those of Euclidean space in each dimension. In [14], Epstein and Thurston proved that the special linear group $SL_n(\mathbb{Z})$ is *not* automatic for $n \geq 3$ by showing that $SL_n(\mathbb{Z})$ satisfies an *exponential* isoperimetric inequality in high dimensions. If restricted to low dimensions, one would find that the isoperimetric function (measuring fillings of loops with disks) for $SL_n(\mathbb{Z})$ is *quadratic* for $n \geq 5$ [35]. This could falsely be interpreted as evidence that $SL_n(\mathbb{Z})$ is automatic.

1.3 Statement of Main Result

Recently, the study of non–positively curved groups has become an extremely active area of research and many important groups in Geometric Group Theory arise as subgroups of non–positively curved groups. These subgroups act on a subspace of a non–positively curved space and are commonly studied by investigating the subspace it acts on. Therefore it is desirable to know which properties of the ambient space are inherited by the subspace.

We will investigate the relationship between isoperimetric functions of a subspace and ambient space. Like the examples in Section 1.1, these functions can be used as group–theoretic invariants. Our main result gives sufficient conditions (in terms of finiteness properties) for a (homological) isoperimetric function of a subgroup to be bounded above by that of the ambient group; see Chapter 3.1 for definitions. This contrasts with known examples illustrating that this relationship does not always hold when our conditions are relaxed; see Chapter 5.2 for examples.

Theorem 1.3.1 (Main Result) *Let G be a group admitting a finite $(n+1)$-dimensional $K(G,1)$ and let $H \leq G$ be a subgroup of type F_{n+1}. Then $FV_H^{n+1} \preceq FV_G^{n+1}$.*

Remark 1.3.2 *The above result is contained in the paper [21] which has recently been submitted.*

1.4 How to Read this Thesis

A suggested first reading of this thesis is as follows:

- Chapter 2 is included primarily for motivational purposes, and can be omitted if one wishes.
- To understand the *content* of our main result, Theorem 1.3.1, one should first read Chapter 3.1 and have a basic understanding of cellular homology. Known examples which contrast Theorem 1.3.1 are then provided in Chapter 5.2.
- To understand the applications of Theorem 1.3.1 in Chapters 5.3 and 5.4, one should first read Chapters 3.2 and 3.3.
- To understand the *proof* of Theorem 1.3.1, Chapter 4 must first be read. A working knowledge of Algebraic Topology and Homological Algebra is also required, for which the reader can consult Chapter 8.
- To gain a more intuitive feel of the functions we are studying, the reader is encouraged to read Chapter 6 for examples of various low–dimensional isoperimetric functions.

Chapter 2

Gromov's Filling Theorem

2.1 The Complexity of the Word Problem

2.1.1 The Word Problem

Given a set S we define $S^{-1} = \{ \ s^{-1} \mid s \in S \ \}$ to be the set of *formal inverses* of S. The union $S \cup S^{-1}$ is an *alphabet* and a *word* ω over the alphabet $S \cup S^{-1}$ is a *finite* string of elements, each in $S \cup S^{-1}$. We allow ω to be the empty string consisting of no elements, in which case ω is referred to as the *empty word* and is represented by e. A word $\omega = s_1 s_2 \ldots s_n$ over $S \cup S^{-1}$ is *freely reduced* if $s_{i+1} \neq s_i^{-1}$ for all $1 \leq i \leq n-1$, and the set of all freely reduced words over $S \cup S^{-1}$ is denoted by $F(S)$. If a word ω is not freely reduced, then we may obtain a freely reduced word

ω' by inductively removing subwords in ω of the form $s_i s_i^{-1}$. Notice that once the initial redundancies in ω are removed, new ones may arise. Therefore our reduction algorithm may not terminate after just one round of reducing ω; this is illustrated in Example 2.1. A somewhat subtle fact is that the freely reduced word obtained by this reduction algorithm is *unique* (Bowditch provides a nice explanation of this fact on page 15 of his notes [6]). We will write " = " when two words ω_1 and ω_2 have the same freely reduced word in $F(S)$.

Example 2.1 *Freely reducing a word to the empty word:*

$$\begin{aligned}\omega &= s_1 s_2 s_2^{-1} s_1^{-1} s_2 s_1 s_1^{-1} s_2^{-1} \\ &= s_1 s_1^{-1} s_2 s_2^{-1} \\ &= e\end{aligned}$$

We now turn our attention to the case that $(G, \cdot)$ is a group and S is a subset of G. For a word $\omega = s_1 s_2 \ldots s_n$ over $S \cup S^{-1}$, we write $\omega \equiv g$ if the the product $s_1 \cdot s_2 \cdots s_n = g$ (assuming the natural interpretation of $S \cup S^{-1}$ as elements of G).

Definition 2.1.1 (Generating Set) *Let G be a group. A subset $S \subseteq G$ is a* generating set *for G, denoted by $G = \langle\ S\ \rangle$, if for every $g \in G$, there exists a word ω over $S \cup S^{-1}$ such that $\omega \equiv g$. A group G is* finitely generated *if it has a finite generating set.*

Let $G = \langle\, S \,\rangle$ be a finitely generated group. The *Word problem* for G asks whether or not a word ω over $S \cup S^{-1}$ represents identity in G. A *solution* to the word problem for G is an algorithm that takes such a word ω as input, and outputs whether or not $\omega \equiv 1_G$. It is an easy exercise to show that if the word problem for G is solvable for a particular finite generating set, then it is solvable for any finite generating set of G. Thus it makes to sense to talk about the word problem for G without making reference to a specific finite generating set.

The importance of the Word problem (and other group–theoretic decision problems) was brought into focus by Max Dehn who realized its consequences in low–dimensional topology [12]. In general, the Word problem is unsolvable – the first examples of *finitely presented* groups with unsolvable word problem are due to Boone [5] and Novikov [27]. However, there are many classes of groups for which the Word problem is solvable – Coxeter groups, hyperbolic groups, one relator groups, polycyclic groups, and others; see [25]. If known to be solvable for a particular group, it is natural to ask about the *complexity* of the Word problem.

2.1.2 The Dehn Function

Let G be a group and S a generating set for G. We will refer to words $\omega \equiv 1_G$ which represent the identity in G as *relators*. If we are given a collection of relators R (in terms of the generating set S) for G, then we may use relators in R to prove that other words ω are also relators (see [9, Chapter 1.1] for a more detailed description of this process).

Example 2.2 *Let G be a group generated by 4 elements a, b, c, d such that the product of commutators $r = [a, b][c, d]$ is a relator. Using r, we deduce that the word $\omega = b[a, b^{-1}]b^{-1}[d, c]$ is also a relator.*

$$\begin{aligned}
\omega &= b[a, b^{-1}]b^{-1}[d, c] \\
&= b(ab^{-1}a^{-1}b)b^{-1}(dcd^{-1}c^{-1}) \\
&= b(ab^{-1}a^{-1})(bb^{-1})(dcd^{-1}c^{-1}b)b^{-1} \\
&= b(ab^{-1}a^{-1})(aba^{-1})b^{-1} \\
&= e
\end{aligned}$$

Definition 2.1.2 (Group Presentation) *Let G be a group. Let S be a generating set and let R a set of relators for G. The pair (S, R) is a* presentation *for G, denoted by $G = \langle\ S \mid R\ \rangle$, if every word $\omega \equiv 1_G$ can be shown to be a relator using only elements of R. A group G is* finitely presented *if there exists a presentation $\langle\ S \mid R\ \rangle$ for G such that both S and R are finite sets.*

Remark 2.1.3 *If $\langle\ S \mid R\ \rangle$ is a presentation for a group G, then G is naturally isomorphic to $F(S)/\langle\langle\ R\ \rangle\rangle$ where $F(S)$ is the free group with free basis S and $\langle\langle\ R\ \rangle\rangle$*

is the normal closure of the subgroup $\langle\ R\ \rangle \leq F(S)$.

Definition 2.1.4 (Area of a word) *Let* $\mathcal{P} = \langle\ S \mid R\ \rangle$ *be a finite presentation for a group* G. *Let* ω *be a word over* $S \cup S^{-1}$ *such that* $\omega \equiv 1_G$. *The* area *of* ω *is the minimal amount times a relator in* R *must be applied to* ω *to prove that* $\omega \equiv 1_G$; *that is*

$$\mathsf{Area}_{\mathcal{P}}(\omega) = \min\left\{\ n\ \middle|\ \omega = \prod_{i=1}^{n} \omega_i r_i \omega_i^{-1},\ \textit{where } \omega_i \in F(S),\ r_i \in R \cup R^{-1}\ \right\},$$

where the equality $\omega = \prod_{i=1}^{n} \omega_i r_i \omega_i^{-1}$ *is taken in* $F(S)$.

Definition 2.1.5 (Dehn function) *Let* $\mathcal{P} = \langle\ S \mid R\ \rangle$ *be a finite presentation for a group* G. *The* Dehn function *of* G *for the presentation* $\mathcal{P}$ *is the function* $\delta_{G,\mathcal{P}} : \mathbb{N} \to \mathbb{N}$ *given by*

$$\delta_{G,\mathcal{P}}(n) = \max\left\{\ \mathsf{Area}_{\mathcal{P}}(\omega) \mid \omega \equiv 1_G,\ \textit{length}(\omega) \leq n\ \right\}.$$

For $\delta_{G,\mathcal{P}}$ to be a $\mathbb{N} \to \mathbb{N}$ function, we must show that the maximum in Definition 2.1.5 is a finite number.

Lemma 2.1.6 *Let* $\mathcal{P} = \langle\ S \mid R\ \rangle$ *be a finite presentation for a group* G. *Then* $\delta_{G,\mathcal{P}}(n)$ *is finite for all* $n \in \mathbb{N}$.

Proof Fix $n \in N$. Since S is finite, there are only a finite number of words of length $\leq n$. Take $\omega \equiv 1_G$ to be of length $\leq n$. Then (by Remark 2.1.3) $\omega \in \langle\langle R \rangle\rangle$, meaning that $\omega = \prod_{i=1}^{k} \omega_i r_i \omega_i^{-1}$, where k is finite and each $\omega_i \in F(S),\ r_i \in R \cup R^{-1}$. Therefore $\delta_{G,\mathcal{P}}(n) < \infty$.

For $n \in \mathbb{N}$, the evaluation of the Dehn function $\delta_{G,\mathcal{P}}(n)$ is a least upper bound for the number of times a relator in R must be applied to prove that an arbitrary word $\omega \equiv 1_G$ of length at most n is in fact a relator. While different presentations for the same group may result in different Dehn functions, we claim these differences are only minor. In order to make this claim concrete, we introduce an equivalence relation on non–decreasing functions $f : \mathbb{N} \to \mathbb{N}$ focusing on asymptotic growth.

Definition 2.1.7 (Linearly Equivalent Functions) *Let f and g be non–decreasing functions from $\mathbb{N}$ to $\mathbb{N}$. Define $f \preceq g$ if there exists $C > 0$ such that for all $n \in \mathbb{N}$*

$$f(n) \leq Cg(Cn + C) + Cn + C.$$

The functions f and g are linearly equivalent*, written $f \sim g$, if both $f \preceq g$ and $g \preceq f$ hold. It is routine to show that $\sim$ is an equivalence relation. The equivalence class containing a function f is called the* linear equivalence class of f.

Observe that if $p \colon \mathbb{N} \to \mathbb{N}$ is a polynomial of degree $d \geq 2$, then the linear equivalence class of p contains all polynomials of degree d. If p is of degree 0 or 1, then the linear equivalence class of p contains all linear functions *and* all constant functions.

When referring to the Dehn function of G, δ_G, without mention of a specific presentation, we simply mean the linear equivalence class of $\delta_{G,\mathcal{P}}$ for an arbitrary finite presentation of G. The following proposition asserts that δ_G is independent of the choice of $\mathcal{P}$ up to the equivalence relation $\sim$, hence δ_G is an *invariant* of the group G.

Proposition 2.1.8 *[9, Prop 1.3.3] Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be finite presentations for a group G. Then $\delta_{G,\mathcal{P}_1} \sim \delta_{G,\mathcal{P}_2}$.*

By definition, δ_G can be interpreted as a measure of complexity of the word problem for G; this is the content of Theorem 2.1.9 below.

Theorem 2.1.9 (Gersten) *[17] Let G be a finitely presented group, then G has solvable word problem if and only if its Dehn function δ_G is a computable function.*

2.2 The Geometry of Least–area Disks

2.2.1 Plateau's Problem

Given a simple closed curve ℓ of finite length in $\mathbb{R}^3$, a *least–area disk* for ℓ is a disk D with boundary ℓ of minimal area. By replicating such a curve with some wire, one can obtain a physical model of a least–area disk by dipping the wire into a soap solution resulting in what is called a *soap film.* The question whether or not every simple closed curve of finite length in 3-dimensional Euclidean space bounds a disc (or more generally a surface) of minimal area is known as *Plateau's problem* in honour of Belgian physicist Joseph Plateau who did pioneering work on the structure of soap films [29].

Plateau's problem was first solved by Douglas [13] and Radó [30], and later generalized by Morrey [26] to the case where $\mathbb{R}^3$ can be replaced with the universal cover $\widetilde{M}$ of any closed, smooth Riemannian manifold M. Roughly speaking, an n–dimensional Riemannian manifold M is a topological space locally homeomorphic to n-dimensional Euclidean space equipped with a nice metric allowing M to be studied using tools from multi-variable calculus. If one knows that least–area disks exist, it is natural to ask questions about their geometry such as: *can the area of an arbitrary least–area disc be bounded by a function f of the length of it's boundary?*

Such a function f is called an *isoperimetric function* as it must be similar in nature to the well–known isoperimetric inequality of the Euclidean place $\mathbb{E}$: for any simple closed curve ℓ in $\mathbb{E}$ of length L, the area of the region ℓ encloses is bounded above by $\frac{L^2}{4\pi}$ with equality holding only when ℓ is a circle.

2.2.2 Gromov's Filling Theorem

A function between metric spaces $f : X \to Y$ is *Lipschitz* if there exists a constant $C > 0$ such that $d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.

Definition 2.2.1 *Let $\widetilde{M}$ be the universal cover of closed, smooth Riemannian manifold M and let $\ell : S^1 \to \widetilde{M}$ be a loop of finite length. The* (genus 0) filling area *of ℓ is given by*

$$FArea(\ell) = \inf\{\ \mathsf{Area}(f) \mid f : D^2 \to \widetilde{M} \text{ is Lipschitz, } f|_{\partial(D^2)} = \ell\ \}.$$

Definition 2.2.2 *The genus zero, 2-dimensional filling function of $\widetilde{M}$ is the function $Fill^{\,0}_{\widetilde{\mathrm{M}}} : [0, \infty) \to [0, \infty)$ given by*

$$Fill^{\,0}_{\widetilde{\mathrm{M}}}(L) = \sup\{\ FArea(\ell) \mid length(\ell) \leq L\ \}.$$

For $L \in [0, \infty)$, the evaluation of the filling function $\mathrm{Fill}^{\,0}_{\widetilde{\mathrm{M}}}(L)$ is a least upper bound for the minimal area required to fill an arbitrary loop $\ell \colon S^1 \to \widetilde{M}$ of length at most L with a disk. Now that we have defined the functions $\mathrm{Fill}^{\,0}_{\widetilde{\mathrm{M}}}$ and δ_G, we are in a position to state Gromov's Filling Theorem.

Theorem 2.2.3 (Gromov's Filling Theorem) *[9, Theorem 2.1.2] Let M be a closed, smooth Riemannian manifold and let $\widetilde{M}$ be its universal cover. Then $Fill^{\,0}_{\widetilde{\mathrm{M}}} \simeq \delta_{\pi_1(M)}$.*

What makes Theorem 2.2.3 so remarkable is that it relates the *complexity* of a group–theoretic decision problem for $\pi_1(M)$ – to an analytic problem concerning the *geometry* of the Riemannian manifold $\widetilde{M}$. Using this connection, one can use group–theoretic properties of $\pi_1(M)$ to prove geometric properties of $\widetilde{M}$ and vice–versa; see [9] for a more in–depth discussion of Theorem 2.2.3 and Dehn functions in general.

Remark 2.2.4 (The equivalence relation $\simeq$ in Theorem 2.2.3) *In Theorem 2.2.3, we take $\simeq$ to be the following natural extension of Definition 2.1.7 to non–decreasing $[0,\infty) \to [0,\infty)$ functions: any $\mathbb{N} \to \mathbb{N}$ function f is extended to a $[0,\infty) \to [0,\infty)$ function by defining f to be constant on the interval $[n, n+1)$ for all $n \in \mathbb{N}$. However, for the rest of this thesis we will only be concerned with $\mathbb{N} \to \mathbb{N}$ functions.*

Remark 2.2.5 (Finiteness of Fill $^0_{\widetilde{\mathrm{M}}}$) *For Fill $^0_{\widetilde{\mathrm{M}}}$ to be a $[0,\infty) \to [0,\infty)$ function we must show that the supremum in Definition 2.2.2 is finite for all values of L. A proof using upper bounds on the sectional curvature of M can be found in [9, Lem 2.1.4]. Alternatively, by showing that Fill $^0_{\widetilde{\mathrm{M}}} \preceq \delta_{\pi_1(M)}$ (the easier part of Theorem 2.2.3) one can use Lemma 2.1.6 to conclude that Fill $^0_{\widetilde{\mathrm{M}}}(L)$ is finite for all values of L.*

Chapter 3

Generalized Isoperimetric Functions

In this chapter we introduce two types of filling functions for topological spaces X of an isoperimetric nature similar to the function $\mathrm{Fill}\,^{0}_{\widetilde{\mathrm{M}}}$ defined in Chapter 2:

- *Homological* filling functions FV_X^{n+1} where we consider fillings of n–cycles by $(n+1)$–chains, and
- *Homotopical* filling functions δ_X^n where we consider fillings of n–spheres by $(n+1)$–balls.

3.1 Homological Filling Functions FV_X^{n+1}

We first recall Definition 2.1.7 from Chapter 1 as it will be used regularly throught the remainder of this thesis:

Definition 3.1.1 (Linearly Equivalent Functions) *Let f and g be non–decreasing functions from $\mathbb{N}$ to $\mathbb{N}$. Define $f \preceq g$ if there exists $C > 0$ such that for all $n \in \mathbb{N}$*

$$f(n) \leq Cg(Cn + C) + Cn + C.$$

The functions f and g are linearly equivalent, *written $f \sim g$, if both $f \preceq g$ and $g \preceq f$ hold. It is routine to show that $\sim$ is an equivalence relation. The equivalence class containing a function f is called the* linear equivalence class of f.

Let X be a cell complex. The *cellular chain group* $C_i(X)$ is the free Abelian group with basis the collection of all i–cells in X. If $\alpha = \sum n_i \Delta_i$ is a cellular chain in X with $n_i \in \mathbb{Z}$ and Δ_i each distinct i–cells of X, then we define the ℓ_1*–norm* (see Definition 4.1.1 and Example 4.1) of α to be $\|\alpha\|_1 = \sum |n_i|$.

Definition 3.1.2 ((Cellular) n^{th}–homological filling function) *Let X be an n–connected cell complex. The* filling volume *of a cellular n–cycle $\alpha \in Z_n(X)$ is given by*

$$FVol_X^{cell}(\alpha) = \min\{\ \|\beta\|_1 \mid \beta \in C_{n+1}(X),\ \partial(\beta) = \alpha\ \}.$$

The (cellular) n^{th}–homological filling function *of* X *is the function* $FV_X^{n+1}\colon \mathbb{N} \to \mathbb{N}$ *given by*

$$FV_X^{n+1}(k) = \max\{\ FVol_X^{cell}(\alpha) \mid \alpha \in Z_n(X), \|\alpha\|_1 \leq k\ \}.$$

Recall that a $K(G,1)$ for a given group G is a space X with $\pi_1(X) \simeq G$ such that the universal cover $\widetilde{X}$ is contractible. It is well known that every group admits a $K(G,1)$; see Chapter 8.1.

Definition 3.1.3 (F_n group) *[11] A group G is of type F_n if G admits a $K(G,1)$ with finite n–skeleton, i.e., only finitely many cells in dimensions $\leq n$.*

Remark 3.1.4 *[15] Let G be a group. Then G is of type F_1 if and only if G is finitely generated, and is of type F_2 if and only if G is finitely presented.*

Definition 3.1.5 (n^{th}–homological filling function of a group) *Let $n \geq 1$ and let G be a group of type F_{n+1}. Let X be a $K(G,1)$ with finite $(n+1)$–skeleton. The (cellular)* n^{th}–homological filling function *of G is defined as $FV_G^{n+1} = FV_{\widetilde{X}}^{n+1}$, where $\widetilde{X}$ is the universal cover of X.*

Theorem 3.1.6 (FV_G^{n+1} is a Well–defined Group invariant) *[34, Lemma 1] Let G be a group of type F_{n+1}, $n \geq 1$. Then the linear equivalence class of FV_G^{n+1} is independent of choice of $K(G,1)$.*

Remark 3.1.7 *It is not immediately clear that the maximum in Definition 3.1.2 is a finite number. In Chapter 3.4 we recall some results from the literature which imply that* FV_G^{n+1} *is a finite–valued function for* $n \geq 3$. *The case* $n = 1$ *can be found in [16, Prop. 2.4]. I am not aware of a proof for the case* $n = 2$.

For $n = 2$, *all results regarding* FV_G^3 *hold under the following natural modifications: First, work with the standard extensions of addition, multiplication, and order, of the positive integers* $\mathbb{N}$ *to* $\mathbb{N} \cup \{\infty\}$. *Definition 2.1.7 is extended to non–decreasing functions* $\mathbb{N} \to \mathbb{N} \cup \{\infty\}$, *but we emphasize that the constant* C *remains a finite positive integer. In Definition 3.1.2 the function* FV_G^3 *is defined as an* $\mathbb{N} \to \mathbb{N} \cup \{\infty\}$ *function. We remark that all arguments in this thesis do not rely on finiteness of* $FV_G^{n+1}(k)$.

3.2 Homotopical Filling Functions δ_X^n

We will use the following definition of Brady et al. [8] to define some higher–dimensional filling functions for cell complexes. Details concerning admissible maps (such as their existence) can be found in [8].

Definition 3.2.1 (Admissible map) *[8] Let* X *be a cell complex and let* M *be a compact* n*-manifold (possibly with boundary). A continuous map* $f\colon M \to X^{(n)}$

is admissible *if* $f^{-1}(X^{(n)} - X^{(n-1)})$ *is a disjoint union of open* n*–balls in* M*, each of which is mapped homeomorphically to an* n*–cell of* X*. The volume* $vol(f)$ *is the number of these* n*–balls.*

Definition 3.2.2 (n^{th}–homotopical filling function) *[8] Let* $n \geq 1$ *and let* X *be an* n*–connected cell complex. The* filling volume *of an admissible map* $\alpha\colon S^n \to X$ *is given by*

$$FVol_X^{adm}(\alpha) = \min\{\ vol(\beta) \mid \beta\colon D^{n+1} \to X,\ \beta|_{\partial(D^{n+1})} = \alpha,\ \beta \text{ is admissible }\}.$$

The n^{th}–homotopical filling function *of* X *is the function* $\delta^n_X\colon \mathbb{N} \to \mathbb{N}$ *given by*

$$\delta^n_X(k) = \max\{\ FVol_X^{adm}(\alpha) \mid \alpha\colon S^n \to X \text{ is admissible, } vol(\alpha) \leq k\ \}.$$

Definition 3.2.3 (n^{th}–homotopical filling function of a group) *Let* $n \geq 1$ *and let* G *be a group of type* F_{n+1}*. Let* X *be a* $K(G, 1)$ *with finite* $(n+1)$*–skeleton. The* n^{th}–homotopical filling function *of* G *is defined as* $\delta^n_G = \delta^n_X$*.*

Theorem 3.2.4 (δ^n_G is a Well–defined Group invariant) *[3, Corollary 1] Let* G *be a group of type* F_{n+1}*,* $n \geq 1$*. Then the linear equivalence class of* δ^n_G *is independent of choice of* $K(G, 1)$*. Moreover,* δ^n_G *is a finite–valued function.*

Remark 3.2.5 (Dehn Function) *If* G *is finitely presented, then* δ^1_G *is referred to as the* Dehn function *of* G *and is denoted by* δ_G*.*

Remark 3.2.6 *The approach that Alonso et al. use in [3] to define* $\delta^n_{\widetilde{X}}$ *is more algebraic than Definition 3.2.2 and involves using higher homotopy groups* $\pi_k(X)$ *as* $\pi_1(X)$*–modules. It is observed in [8, Remark* 2.4(2)*] that these two definitions are equivalent.*

Remark 3.2.7 *We require* $n \geq 1$ *for all our filling functions. For our purposes, it does not make sense consider filling* 0*–spheres by* 1*–balls as the maximum in Definition 3.2.2 would not be attained; if* X *is not compact, then two points can be an unbounded distance apart.*

3.3 Some Recent Results

Since the function FV^{n+1}_X places no restriction on the topology of $(n+1)$–chains used to fill n–cycles in X, while δ^n_X restricts to filling n–spheres with $(n+1)$–balls, it would seem possible for FV^{n+1}_X and δ^n_X to be quite different. However, when n is large enough, the Hurewicz Theorem (see Theorem 7.1.2) can be used to show that $FV^{n+1}_X \sim \delta^n_X$.

Theorem 3.3.1 (Relation between FV^{n+1}_G and δ^n_G) *[1] The* n^{th}*–homological and homotopical filling functions* FV^{n+1}_G *and* δ^n_G *are linearly equivalent for* $n \geq 3$*. When* $n = 2$*,* $FV^3_G \succeq \delta^2_G$*.*

Theorem 3.3.2 (Non-equivalence of δ_G^n and FV_G^{n+1} in Low Dimensions) *There exists groups G with $FV_G^2 \precsim\!\!\!\!\not\;\;\approx \delta_G^1$ and groups H with $FV_H^3 \succsim\!\!\!\!\not\;\;\approx \delta_H^2$.*

Definition 3.3.3 (Hyperbolic Group) *A group G is* hyperbolic *if it has a linear Dehn function δ_G^1.*

Theorem 3.3.4 *[23] Let G be a hyperbolic group. Then FV_G^{n+1} is linear for all $n \geq 1$.*

Corollary 3.3.5 *Let G be a hyperbolic group. Then δ_G^n is linear for all $n \geq 1$.*

3.4 Finiteness of FV_G^{n+1}

Let G be a group of type F_{n+1} We will sketch why FV_G^{n+1} is a finite valued function for $n \geq 3$.

Finiteness of FV_G^{n+1} follows from Theorem 3.2.4 and the inequality

$$FV_G^{n+1} \preceq \delta_G^n,$$

which holds for all $n \geq 3$. We outline the argument for this inequality described in the introduction of [1]: Let X be a $K(G,1)$ with finite $(n+1)$–skeleton and let $\widetilde{X}$ be its universal cover. Take $\gamma \in Z_n\left(\widetilde{X}\right)$ with $\|\gamma\|_1 \leq k$. By using the Hurewicz Theorem, one can show (see [19, 32]) that γ is the image of the fundamental class of an n–sphere for an admissable map $f\colon S_n \to \widetilde{X}$ such that $vol(f) = \|\gamma\|_1$. If $\tilde{f}\colon D^{n+1} \to \widetilde{X}$ is an extension of f to the $(n+1)$–ball D^{n+1}, then the image of the fundamental class of D^{n+1} is a cellular $(n+1)$–chain $\tilde{f}([D^{n+1}])$ with $\partial\left(\tilde{f}([D^{n+1}])\right) = \gamma$ and $vol\left(\tilde{f}\right) \geq \|\tilde{f}([D^{n+1}])\|_1$ (see Remark 3.4.2). Consequently, the admissible filling volume $\mathrm{FVol}^{adm}_{\widetilde{X}}(f)$ is greater than or equal to the cellular filling volume $\mathrm{FVol}^{cell}_{\widetilde{X}}(\gamma)$. It follows from Theorem 3.2.4 that $FV_G^{n+1}(k) \leq \delta_G^n(k) < \infty$.

Remark 3.4.1 *This approach does not work for the case $n = 2$ as there are groups with $FV_G^3 \succnsim \delta_G^2$; see Appendix 6.4.*

Remark 3.4.2 (Relationship between FV_X^{n+1} and δ_X^n) *If M is a compact orientable k–manifold (with or without boundary) and $\alpha\colon M \to X$ is an admissible map, then the image of the fundamental class of M, denoted by $\alpha([M])$, is a cellular k–chain of X. It may happen that two k–balls in M are mapped to the same k–cell of X but with opposite orientations, in which case the two k-balls of M cancel in the induced k–chain $\alpha([M])$. Therefore $FVol_X^{cell}(\alpha([S^n]) \leq FVol_X^{adm}(\alpha)$. This inequality may lead one to believe that $FV_X^{n+1} \preceq \delta_X^n$. However, it is important to realize that FV_X^{n+1} places no restrictions on the topology of fillings for cycles, while δ_X^n restricts to using balls. It may happen that there are closed compact n–manifolds mapping to X which require substantially more volume to fill than n–spheres, in which case $FV_X^{n+1} \succeq \delta_X^n$; see Chapter 6.4 for an example.*

Chapter 4

Filling Norms on $\mathbb{Z}G$–modules

Based on ideas of Gersten [18], we introduce the notion of a *filling norm* on a $\mathbb{Z}G$–module which serves as an algebraic analogue of the various filling volumes considered in Chapter 3. In particular, we give an algebraic definition of FV_G^{n+1}, prove that FV_G^{n+1} is a well–defined invariant of G, and show that our algebraic definition of FV_G^{n+1} is equivalent to the topological definition given in Chapter 3.

4.1 Filling Norms on $\mathbb{Z}G$-Modules

Definition 4.1.1 (Norm on an Abelian Group) *A* norm *on an abelian group A is a function $\|\cdot\|\colon A \to \mathbb{R}$ satisfying the following conditions:*

- $\|a\| \geq 0$ *with equality if and only if* $a = e$*,*
- $\|a\| + \|a'\| \geq \|a + a'\|$,
- $\|na\| = |n| \cdot \|a\|$*, where* $n \in \mathbb{Z}$.

Example 4.1 (ℓ_1–norm) *If A is a free abelian group with basis X, then X induces an* ℓ_1–norm *on A given by* $\left\| \sum_{x \in X} n_x x \right\|_1 = \sum_{x \in X} |n_x|$*, where* $n_x \in \mathbb{Z}$.

Definition 4.1.2 (Linearly Equivalent Norms) *Two norms $\|\cdot\|$ and $\|\cdot\|'$ on a $\mathbb{Z}$–module M are* linearly equivalent *if there exists a fixed constant $C > 0$ such that*

$$C^{-1}\|m\| \leq \|m\|' \leq C\|m\|$$

for all $m \in M$. This is an equivalence relation and the equivalence class of a norm $\|\cdot\|$ is called the linear equivalence class of $\|\cdot\|$.

Definition 4.1.3 (Based Free $\mathbb{Z}G$–modules and Induced ℓ_1-norms) *Suppose G is a group and F is a free $\mathbb{Z}G$–module with finite $\mathbb{Z}G-$basis $\{\alpha_1, \ldots, \alpha_n\}$. Then the set $\{g\alpha_i\colon g \in G, 1 \leq i \leq n\}$ is a free $\mathbb{Z}$-basis for F as a $\mathbb{Z}$–module. This free $\mathbb{Z}$–basis* *induces an ℓ_1-norm $\|\cdot\|_1$ on F. We call a free $\mathbb{Z}G-$module* based *if it is understood to have a fixed basis, and we use this basis for the induced ℓ_1–norm $\|\cdot\|_1$.*

Definition 4.1.4 (Filling Norms on $\mathbb{Z}G$-modules) *Let $\eta\colon F \to M$ be a surjective homomorphism of $\mathbb{Z}G$-modules and suppose that F is free, finitely generated, and based. The* filling norm on M induced by η and the free $\mathbb{Z}G$-basis of F *is defined as*

$$\|m\|_\eta = \min\{\ \|x\|_1 \colon x \in F,\ \eta(x) = m\}\,.$$

Remark 4.1.5 (Induced ℓ_1-norms are Filling Norms) *If F is a finitely generated based free $\mathbb{Z}G$-module, then the ℓ_1-norm induced by a free $\mathbb{Z}G$-basis is a filling norm (it is equal to the filling norm induced by the identity map $F \to F$).*

Remark 4.1.6 (Filling norms on $\mathbb{Z}G$–modules are G–equivariant) *In Definition 4.1.3, the group G acts freely on the free $\mathbb{Z}$–basis $\{g\alpha_i \colon g \in G, 1 \leq i \leq n\}$ by left multiplication. It follows that the induced ℓ_1–norm $\|\cdot\|_1$ on F is G–equivariant, meaning that for any $x \in F$, $g \in G$ we have $\|g.x\|_1 = \|x\|_1$. Combining this fact with Definition 4.1.4 shows that filling norms on finitely generated $\mathbb{Z}G$–modules are G–equivariant.*

The following lemma is reminiscent of the fact that linear operators on finite dimensional normed spaces are bounded.

Lemma 4.1.7 ($\mathbb{Z}G$-Morphisms between Free Modules are Bounded) *[18, Lemma 4.1] Let $\varphi : F \to F'$ be a homomorphism between finitely generated, free, based $\mathbb{Z}G-$modules. Let $\|\cdot\|_1$ and $\|\cdot\|_1'$ denote the induced ℓ_1–norms of F and F'. Then there exists a constant $C > 0$ such that for all $x \in F$*

$$\|\varphi(x)\|_1' \leq C \cdot \|x\|_1.$$

Proof Let $A = \{\alpha_1, \ldots, \alpha_n\}$ be the $\mathbb{Z}G-$basis of F inducing the norm $\|\cdot\|_1$. Then φ is given by a finite $n \times m$ matrix whose entries are elements of $\mathbb{Z}G$. Define $C = \max\limits_{1\leq i\leq n} \{ \ \|\varphi(\alpha_i)\|_1' \ \}$ and let $x \in F$ be arbitrary. Then

$$\begin{aligned}
\|\varphi(x)\|_1' &= \|\varphi(\lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n)\|_1' \\
&\leq \left\| \left(\sum_{g\in G} \lambda_{1,g} g\right) \varphi(\alpha_1)\right\|_1' + \cdots + \left\|\left(\sum_{g\in G} \lambda_{n,g} g\right) \varphi(\alpha_n) \ \right\|_1' \\
&\leq \left(\sum_{g\in G} |\lambda_{1,g}|\right) \|\varphi(\alpha_1)\|_1' + \cdots + \left(\sum_{g\in G} |\lambda_{n,g}|\right) \|\varphi(\alpha_n)\|_1' \\
&\leq C \left(\sum_{i=1}^{m} \left(\sum_{g\in G} |\lambda_{i,g}|\right)\right) = C\|x\|_1.
\end{aligned}$$

where $\lambda_i \in \mathbb{Z}G$ and $\lambda_j = \sum\limits_{g\in G} \lambda_{j,g} g, \ \lambda_{j,g} \in \mathbb{Z}$.

Lemma 4.1.8 ($\mathbb{Z}G$-Morphisms with Projective Domain are Bounded) *Let $\varphi\colon P \to Q$ be a homomorphism between finitely generated $\mathbb{Z}G-$modules. Let $\|\cdot\|_P$ and $\|\cdot\|_Q$ denote filling norms on P and Q respectively. If P is projective then there* *exists a constant $C > 0$ such that for all $p \in P$*

$$\|\varphi(p)\|_Q \leq C \cdot \|p\|_P.$$

Proof Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\tilde{\varphi}"] \arrow[d, "\rho"'] & B \arrow[d] \\ P \arrow[r, "\varphi"'] \arrow[ur, "\psi"] & Q \end{tikzcd}$$

constructed as follows. Let A and B be finitely generated and based free $\mathbb{Z}G$-modules, and let $A \to P$ and $B \to Q$ be surjective morphisms inducing the filling norms $\| \cdot \|_P$ and $\| \cdot \|_Q$. Since P is projective and $B \to Q$ is surjective, there is a lifting $\psi \colon P \to B$ of φ; then let $\tilde{\varphi}$ be the composition $A \to P \xrightarrow{\psi} B$. Let C be the constant provided by Lemma 4.1.7 for $\tilde{\varphi}$. Let $p \in P$ and let $a \in A$ that maps to p. It follows that

$$\|\varphi(p)\|_Q \leq \|\psi(p)\|_1 = \|\tilde{\varphi}(a)\|_1 \leq C\|a\|_1.$$

Since the above inequality holds for any $a \in A$ with $\rho(a) = p$, it follows that

$$\begin{aligned} \|\varphi(p)\|_Q &\leq C \cdot \min_{\rho(a)=p} \{\|a\|_1\} \\ &= C \cdot \|p\|_P. \end{aligned}$$

Lemma 4.1.9 (Equivalence of Filling Norms for $\mathbb{Z}G$-modules) *[18, Lemma 4.1]* *Any two filling norms on a finitely generated $\mathbb{Z}G$–module M are linearly equivalent.*

Proof Consider a pair of surjective homomorphisms of $\mathbb{Z}G-$modules $\eta : F \to M$ and $\eta' : F' \to M$ such that F and F' are finitely generated, free, based modules inducing the filling norms $\|\cdot\|_\eta$ and $\|\cdot\|_{\eta'}$ on M. Since η' is surjective, the universal property of F provides a homomorphism φ such that $\eta = \eta' \circ \varphi$. Let $C > 0$ be the constant of Lemma 4.1.7 for φ. Let $m \in M$ be arbitrary and take $x \in F$ such that $\eta(x) = m$. Since $\eta' \circ \varphi(x) = m$, we have

$$\|m\|_{\eta'} = \min_{\eta'(x')=m} \|x'\|'_1 \leq \|\varphi(x)\|'_1 \leq C \cdot \|x\|_1.$$

As this inequality holds for all $x \in F$ satisfying $\eta(x) = m$, we have

$$\|m\|_{\eta'} \leq C \cdot \min_{\eta(x)=m} \{\|x\|_1\} = C \cdot \|m\|_\eta.$$

The other inequality proceeds in a similar manner.

Lemma 4.1.10 (Retraction Lemma) *[18, Prop. 4.4] Let* $0 \xrightarrow{\iota} M \to N \to P \to 0$ *be a short exact sequence of* $\mathbb{Z}G-$*modules where*

1) M is finitely generated and equipped with a filling-norm $\|\cdot\|_M$.

2) N is free, based, and equipped with the induced ℓ_1-norm $\|\cdot\|_1$.

3) P is projective.

Then there exists a retraction $\rho : N \to M$ for the inclusion $\iota : M \to N$ and a fixed constant $C > 0$ such that $\|\rho(x)\|_M \leq C\|x\|_1$ for all $x \in N$.

Proof Since P is projective there is a retraction ρ' for the map $\iota\colon M \to N$. Since M is finitely generated, N is isomorphic to a product $I \oplus Q$ of free modules where I is finitely generated and contains the image of M. Define $\rho : N \to M$ by $\rho|_I = \rho'|_I$ and $\rho|_Q = 0$. Then ρ is a retraction for ι with support contained in I.

Each $x \in N$ has a unique decomposition $x = y + q$ where $y \in I,\ q \in Q$ such that $\rho(x) = \rho(y)$ and $\|y\|_1 \leq \|x\|_1$. Apply Lemma 4.1.8 to the restriction $\rho\colon I \to M$ to obtain $C > 0$ such that

$$\|\rho(x)\|_M = \|\rho(y)\|_M \leq C\|y\|_1 \leq C\|x\|_1.$$

4.2 Defining FV_G^{n+1} Algebraically

We will now give an algebraic definition of the invariant FV_G^{n+1} and prove its equivalence with the topological definition (see Definition 3.1.5) in Chapter 4.3.

Definition 4.2.1 (FP_n group) *[11] A group G is of* type FP_n *if there is a resolution of $\mathbb{Z}G$–modules*

$$P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow \mathbb{Z} \to 0,$$

*such that for each $i \in \{0, 1 \dots, n\}$ the module P_i is a finitely generated projective $\mathbb{Z}G$–module. In this case, such a resolution is called an FP_n–*resolution of length *n.*

Remark 4.2.2 *The $\mathbb{Z}G$–module structure of $\mathbb{Z}$ arises from the trivial action of G on $\mathbb{Z}$. See Remark 8.2.11 for details.*

Definition 4.2.3 (Algebraic definition of FV_G^{n+1}) *Let G be a group of type FP_{n+1}. The algebraic n^{th}–*filling invariant *is the (linear equivalence class of the) function*

$$FV_G^{n+1} \colon \mathbb{N} \to \mathbb{N}$$

defined as follows: Let

$$P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow \mathbb{Z} \to 0,$$

be a resolution of $\mathbb{Z}G$–modules for $\mathbb{Z}$ of type FP_{n+1}. Choose filling norms for P_n and P_{n+1}, denoted by $\|\cdot\|_{P_n}$ and $\|\cdot\|_{P_{n+1}}$ respectively. Then

$$FV_G^{n+1}(k) = \max\left\{ \ \|\gamma\|_{\partial_{n+1}} \colon \gamma \in \ker(\partial_n), \ \|\gamma\|_{P_n} \le k \ \right\},$$

where

$$\|\gamma\|_{\partial_{n+1}} = \min\left\{\|\mu\|_{P_{n+1}} \colon \mu \in P_{n+1}, \ \partial_{n+1}(\mu) = \gamma\right\}.$$

Remark 4.2.4 *Proposition 4.1.9 allows us to replace the norm $\|\cdot\|_{\partial_{n+1}}$ in Definition 4.2.3 with any choice of filling norm $\|\cdot\|_{Z_n}$ on $\ker(\partial_n)$. This will not affect the linear equivalence class of the function FV_G^{n+1}.*

Remark 4.2.5 *It is not immediatly clear that the maximum in Definition 4.2.3 is a finite number; in particular Remark 3.1.7 also applies here. Finiteness of FV_G^{n+1} for finitely presented groups of type FP_{n+1} follows from the observations in Chapter 3.4 and Corollary 4.3.3. Up to this point I have been unsuccessful in producing a self–contained algebraic proof of finiteness.*

Theorem 4.2.6 (FV_G^{n+1} is a Well-defined Group Invariant) *Let G be a group of type FP_{n+1}. Then the algebraic n^{th}–filling invariant FV_G^{n+1} of G is well defined up to linear equivalence.*

Proof Let (Q_*, ∂_*) and (P_*, δ_*) be a pair of resolutions of $\mathbb{Z}G$-modules of type FP_{n+1} with choices of filling-norms for their n^{th} and $(n+1)^{\text{th}}$ modules denoted by $\|\cdot\|_{Q_n}$ and $\|\cdot\|_{Q_{n+1}}$ and $\|\cdot\|_{P_n}$ and $\|\cdot\|_{P_{n+1}}$ respectively. Let $FV_{Q_*}^{n+1}$ and $FV_{P_*}^{n+1}$ be the induced functions according to Definition 4.2.3. By symmetry, it is enough to show that $FV_{Q_*}^{n+1} \preceq FV_{P_*}^{n+1}$.

It is well known that any two projective resolutions of a $\mathbb{Z}G$-module are chain homotopy equivalent (see Theorem 8.2.21), hence the resolutions Q_* and P_* are chain homotopy equivalent. This means there exists chain maps $f_i : Q_i \to P_i,\ g_i : P_i \to Q_i$, and $h_i : Q_i \to Q_{i+1}$ such that

$$\partial_{i+1} \circ h_i + h_{i-1} \circ \partial_i = g_i \circ f_i - id.$$

Let C denote the maximum of the constants for the maps g_{n+1}, h_n, and f_n and the chosen filling-norms provided by Lemma 4.1.8. We claim that for every $k \in \mathbb{N}$,

$$FV_{Q_*}^{n+1}(k) \leq C \cdot FV_{P_*}^{n+1}(Ck + C) + Ck + C.$$

Fix $k \in \mathbb{N}$ and let $\alpha \in \ker(\partial_n)$ be such that $\|\alpha\|_{Q_n} \leq k$. Since $f_n(\alpha) \in \ker(\delta_n)$ we may choose $\beta \in P_{n+1}$ such that $\delta_{n+1}(\beta) = f_n(\alpha)$ and $\|f_n(\alpha)\|_{\delta_{n+1}} = \|\beta\|_{P_{n+1}}$. By the chain homotopy equivalence and commutativity of the chain maps we have,

$$\begin{aligned} \partial_{n+1} \circ h_n(\alpha) + h_{n-1} \circ \partial_n(\alpha) &= g_n \circ f_n(\alpha) - \alpha \\ &= g_n \circ \delta_{n+1}(\beta) - \alpha \\ &= \partial_{n+1} \circ g_{n+1}(\beta) - \alpha. \end{aligned}$$

Since $\alpha \in \ker(\partial_n)$, we have that $h_{n-1} \circ \partial(\alpha) = 0$. Rearranging the above equation, we obtain

$$\begin{aligned} \alpha &= \partial_{n+1} \circ g_{n+1}(\beta) - \partial_{n+1} \circ h_n(\alpha) \\ &= \partial_{n+1} \left(g_{n+1}(\beta) - h_n(\alpha) \right). \end{aligned}$$

Therefore $g_{n+1}(\beta) - h_n(\alpha)$ has boundary α. To conclude, observe that

$$\begin{aligned}
\|\alpha\|_{Q_n} &\leq \|g_{n+1}(\beta) - h_n(\alpha)\|_{Q_{n+1}} && \text{since } \partial_{n+1}(g_{n+1}(\beta) - h_n(\alpha)) = \alpha \\
&\leq \|g_{n+1}(\beta)\|_{Q_{n+1}} + \|h_n(\alpha)\|_{Q_{n+1}} && \text{by the triangle inequality} \\
&\leq C \cdot \|\beta\|_{P_{n+1}} + C \cdot \|\alpha\|_{Q_n} && \text{by Lemma 4.1.8} \\
&= C \cdot \|f_n(\alpha)\|_{\delta_{n+1}} + C \cdot \|\alpha\|_{Q_n} && \text{by definition of } \beta \\
&\leq C \cdot FV_{P_*}^{n+1}(\|f_n(\alpha)\|_{P_n}) + C\|\alpha\|_{Q_n} && \text{by definition of } FV_{P_*}^{n+1} \\
&\leq C \cdot FV_{P_*}^{n+1}\left(C\|\alpha\|_{Q_n}\right) + C\|\alpha\|_{Q_n} && \text{by Lemma 4.1.8} \\
&\leq C \cdot FV_{P_*}^{n+1}(Ck + C) + Ck + C && \text{since } \|\alpha\|_{Q_n} \leq k.
\end{aligned}$$

Since α was arbitrary, this shows that $FV_{Q_*}^{n+1} \preceq FV_{P_*}^{n+1}$ completing the proof.

4.3 Equivalence of Topological and Algebraic Definitions for FV_G^{n+1}

We will now proceed to show that the topological and algebraic approaches to FV_G^{n+1} are equivalent for finitely presented groups of type FP_{n+1}.

Proposition 4.3.1 *Let G be a group of type F_{n+1}. Then G is of type FP_{n+1} and the algebraic and topological n^{th}–filling invariants of G are linearly equivalent.*

Proof Let X be a $K(G,1)$ with finite $(n+1)$-skeleton. The G-action on the universal cover $\widetilde{X}$ of X (see Theorem 8.1.10) induces a structure of $\mathbb{Z}G$-module on the group of cellular chains $C_i(\widetilde{X})$ and each boundary map ∂_i is a morphism of $\mathbb{Z}G$-modules. Since the G-action on $\widetilde{X}$ is cellular and free, each $C_i(\widetilde{X})$ is a free $\mathbb{Z}G$-module with basis any collection of representatives of the G-orbits of i-cells. Since the action is cocompact on the $(n+1)$–skeleton, each $C_i(\widetilde{X})$ is a finitely generated free $\mathbb{Z}G$-module for $i \in \{0, 1, \ldots, n+1\}$. Since $\widetilde{X}$ is a contractible space, all its homology groups are trivial and therefore we have a resolution of $\mathbb{Z}G$-modules

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \longrightarrow \mathbb{Z} \to 0,$$

of type FP_{n+1}. Under our assumptions, the induced topological n^{th}–filling invariant of G is a particular instance of an algebraic n^{th}–filling invariant of G. The conclusion follows from Theorem 4.2.6.

Proposition 4.3.2 *[11, pg 205, proof of Thm. 7.1] Let G be finitely presented and of type FP_n where $n \geq 2$. Then G is of type F_n.*

Propositions 4.3.1 and 4.3.2 imply the following statement.

Corollary 4.3.3 *Let G be a finitely presented group of type FP_{n+1}. Then the topological and algebraic definitions of FV_G^{n+1} are equivalent.*

Remark 4.3.4 *Theorem 4.2.6 and Corollary 4.3.3 provide an alternative, self–contained algebraic proof of Theorem 3.1.6. The proof of Theorem 3.1.6 in [34] is a geometric version of the proof of Theorem 4.2.6 presented here and relies on some technical lemmas about cellular quasi-isometries found in [3]. However, we remark that Young's result in [34] is actually more general than Theorem 3.1.6 as his definition of* FV_G^{n+1} *only requires proper actions in place of free actions.*

Chapter 5

A Subgroup Theorem for Homological Filling Functions

In [17, Thm C] Gersten proved the following:

Theorem 5.0.5 (Gersten) *Let G be a group and let $H \leq G$ such that both G and H admit a finite 2–dimensional $K(\cdot, 1)$. Then $FV_H^2 \preceq FV_G^2$.*

Our main result extends Gersten's theorem to higher dimensions:

Theorem 5.0.6 *Let $n \geq 1$. Let G be a group admitting a finite $(n+1)$–dimensional $K(G, 1)$ and let $H \leq G$ is of type F_{n+1}. Then $FV_H^{n+1} \preceq FV_G^{n+1}$.*

We give some examples of homological filling functions illustrating why the above theorem is non–trivial and discuss some applications to hyperbolic groups and homotopical filling functions.

5.1 Main Result

Definition 5.1.1 (Stably finitely generated free) *A $\mathbb{Z}G$-module P is* stably finitely generated free *if there exists finitely generated free $\mathbb{Z}G$ modules F and F' such that $P \oplus F' \simeq F$.*

Lemma 5.1.2 (The Eilenberg Trick) *[11, pg.207] Let $G = \pi_1(X, x_0)$, where X is a cell complex. Then X is a subcomplex of a cell complex Y such that the inclusion $X \hookrightarrow Y$ is a homotopy equivalence and the cellular n-cycles of the universal covers $\widetilde{Y}$ and $\widetilde{X}$ satisfy $Z_n\left(\widetilde{Y}\right) \simeq Z_n\left(\widetilde{X}\right) \oplus \mathbb{Z}G$ as $\mathbb{Z}G$-modules.*

Proof Let x_0 be a 0-cell of X, and glue an n-cell D^n to (X, x_0) by mapping its boundary to x_0. The resulting space is the wedge sum of X and an n-sphere S^n. To obtain Y, attach an $(n+1)$-cell D^{n+1} by the attaching map that identifies ∂D^{n+1} with the n-sphere S^n. It is clear that $X \hookrightarrow Y$ is a homotopy equivalence.

We have that $C_n\left(\widetilde{Y}\right) \simeq C_n\left(\widetilde{X}\right) \oplus \mathbb{Z}G$ where the $\mathbb{Z}G$ factor is generated by a lifting of the n-cell D^n to $\widetilde{Y}$. By definition of the attaching maps, the boundary map $\partial_{\widetilde{Y}} \colon C_n\left(\widetilde{Y}\right) \to C_{n-1}\left(\widetilde{Y}\right)$ is given by $\partial_{\widetilde{X}} \oplus 0$, hence $Z_n\left(\widetilde{Y}\right) \simeq Z_n\left(\widetilde{X}\right) \oplus \mathbb{Z}G$

Lemma 5.1.3 (Schanuel's Lemma) *[11, pg.193, Lemma 4.4] Let*

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

and

$$0 \to P'_n \to P'_{n-1} \to \cdots \to P'_0 \to M \to 0$$

be exact sequences of R–modules with P_i and P'_i projective for $i \leq n-1$. Then

$$P_0 \oplus P'_1 \oplus P_2 \oplus P'_3 \oplus \ldots \;\simeq\; P'_0 \oplus P_1 \oplus P'_2 \oplus P_3 \oplus \ldots$$

We are now ready to prove our main result which is a generalization of [17, Thm C]. The proof is based on Gersten's proof of [18, Thm 4.6] and is adjusted for higher dimensions:

Theorem 5.1.4 *Let $n \geq 1$. Let G be a group admitting a finite $(n+1)$-dimensional $K(G,1)$ and let $H \leq G$ be a subgroup of type F_{n+1}. Then $FV_H^{n+1} \preceq FV_G^{n+1}$.*

Proof Let W be a finite $(n+1)$-dimensional $K(G,1)$. Let X be the $(n+1)$-skeleton of a $K(H,1)$. Since H is of type F_{n+1}, we may assume that X is a finite cell complex. Then (after subdivisions) there exists a cellular map $f : X \to W$ inducing the inclusion $H \hookrightarrow G$ at the level of fundamental groups. Let M_f be the mapping cylinder of f and consider the exact sequences of $\mathbb{Z}G$-modules

$$0 \to Z_n\left(\widetilde{M_f}\right) \to C_n\left(\widetilde{M_f}\right) \to \cdots \to C_0\left(\widetilde{M_f}\right) \to \mathbb{Z} \to 0 \tag{5.1}$$

and

$$0 \to C_{n+1}\left(\widetilde{W}\right) \to C_n\left(\widetilde{W}\right) \to \cdots \to C_0\left(\widetilde{W}\right) \to \mathbb{Z} \to 0, \tag{5.2}$$

where $\widetilde{W}$ and $\widetilde{M_f}$ denote the universal covers of W and M_f respectively.

Applying Schanuel's lemma to the above sequences shows that the $\mathbb{Z}G-$module $Z_n\left(\widetilde{M_f}\right)$ is stably finitey generated free. Let Y be the space obtained by attaching a finite number of $(n+1)$–balls to the base point of M_f as in Lemma 5.1.2 so we can assume that $Z_n\left(\widetilde{Y}\right)$ is finitely generated and free as a $\mathbb{Z}G$-module.

From here on, we are only concerned with the inclusion map $X \to Y$ realizing the inclusion $H \to G$ at the level of fundamental groups, and with the additional property that $Z_n\left(\widetilde{Y}\right)$ is finitely generated and free as a $\mathbb{Z}G$-module. From the first two properties, it follows that any lifting $\widetilde{X} \to \widetilde{Y}$ is an embedding , hence we can assume that $\widetilde{X} \subseteq \widetilde{Y}$. Since the ring $\mathbb{Z}G$ is free as a $\mathbb{Z}H$–module, it follows that $C_i\left(\widetilde{Y}\right)$ is a free a $\mathbb{Z}H$-module. Consequently, the $\mathbb{Z}H-$module $C_i\left(\widetilde{X}\right)$ is a free factor of $C_i\left(\widetilde{Y}\right)$ and the quotient $C_i\left(\widetilde{Y}\right)/C_i\left(\widetilde{X}\right)$ is a free $\mathbb{Z}H-$module.

Restricting our attention to n-skeleta, the following short exact sequence of chain complexes of $\mathbb{Z}H$-modules arises

$$0 \to C_* \left(\widetilde{X}^{(n)}\right) \to C_* \left(\widetilde{Y}^{(n)}\right) \to C_* \left(\widetilde{Y}^{(n)} \ , \ \widetilde{X}^{(n)}\right) \to 0, \tag{5.3}$$

where $C_* \left(\widetilde{Y}^{(n)} \ , \ \widetilde{X}^{(n)}\right)$ is the free quotient complex $C_* \left(\widetilde{Y}^{(n)}\right) / C_* \left(\widetilde{X}^{(n)}\right)$. Consider the induced long exact homology sequence of the pair $(Y^{(n)}, X^{(n)})$:

$$0 \to \widetilde{H}_n \left(\widetilde{X}^{(n)}\right) \to \widetilde{H}_n \left(\widetilde{Y}^{(n)}\right) \to \widetilde{H}_n \left(\widetilde{Y}^{(n)}, \widetilde{X}^{(n)}\right) \to \widetilde{H}_{n-1} \left(\widetilde{X}^{(n)}\right) \to \cdots . \tag{5.4}$$

Since X is the $(n+1)$-skeleton of a $K(H,1)$, the homology group $\widetilde{H}_{n-1}(\widetilde{X}^{(n)})$ is trivial. Now the long exact sequence (5.4) can be truncated, obtaining the short exact sequence

$$0 \to Z_n \left(\widetilde{X}\right) \xrightarrow{\iota} Z_n \left(\widetilde{Y}\right) \to Z_n \left(\widetilde{Y} \ , \ \widetilde{X}\right) \to 0, \tag{5.5}$$

where ι is induced by the inclusion $\widetilde{X} \subseteq \widetilde{Y}$. Note that we have replaced the n–homology module of the n–skeleton by the n–cycles of the full complex. We claim that the short exact sequence (5.5) satisfies the three hypothesis of Lemma 4.1.10.

1) $Z_n \left(\widetilde{X}\right)$ *is finitely generated:* Since X is a finite cell complex, $C_{n+1} \left(\widetilde{X}\right)$ is finitely generated as a $\mathbb{Z}H$-module. Consequently, $Z_n \left(\widetilde{X}\right)$ is also finitely generated as a $\mathbb{Z}H$-module.

2) $Z_n \left(\widetilde{Y}\right)$ *is free:* The construction of Y guarantees that $Z_n \left(\widetilde{Y}\right)$ is a free $\mathbb{Z}G$–module, hence $Z_n \left(\widetilde{Y}\right)$ is a free $\mathbb{Z}H$-module.

3) $Z_n\left(\widetilde{Y},\widetilde{X}\right)$ *is projective:* Restrict attention to n-skeleta and observe that

$$0 \to Z_n\left(\widetilde{Y}^{(n)}\right) \to C_n\left(\widetilde{Y}^{(n)}\right) \to \cdots \to C_0\left(\widetilde{Y}^{(n)}\right) \to \mathbb{Z} \to 0 \tag{5.6}$$

is an exact sequence of $\mathbb{Z}G$-modules, and hence is also an exact sequence of $\mathbb{Z}H$–modules. Analogously, the sequence of $\mathbb{Z}H$–modules

$$0 \to Z_n\left(\widetilde{X}^{(n)}\right) \to C_n\left(\widetilde{X}^{(n)}\right) \to \cdots \to C_0\left(\widetilde{X}^{(n)}\right) \to \mathbb{Z} \to 0 \tag{5.7}$$

is exact. Exactness of (5.6) and (5.7), in addition with the assumption that $\widetilde{X}$ is a subcomplex of $\widetilde{Y}$, implies that the sequence of $\mathbb{Z}H$-modules

$$0 \to Z_n\left(\widetilde{Y}^{(n)},\widetilde{X}^{(n)}\right) \to C_n\left(\widetilde{Y}^{(n)},\widetilde{X}^{(n)}\right) \to \cdots \to C_0\left(\widetilde{Y}^{(n)},\widetilde{X}^{(n)}\right) \to \mathbb{Z} \to 0 \tag{5.8}$$

is also exact. Applying Schanuel's Lemma to the exact sequences (5.6) and (5.8), together with $Z_n\left(\widetilde{Y}^{(n)}\right)$ being a free $\mathbb{Z}H$–module, we conclude that that $Z_n\left(\widetilde{Y},\widetilde{X}\right)$ is a projective $\mathbb{Z}H$–module.

Thus we have shown that the short exact sequence (5.5) satisfies the three hypothesis of Lemma 4.1.10. Before invoking this lemma and concluding the proof, we set up notation for the norms required to specify representatives of FV_G^{n+1} and FV_H^{n+1}.

Let $\|\cdot\|_1$ denote the ℓ_1-norm on $C_i(\widetilde{Y})$ induced by the basis consisting of all i-cells of $\widetilde{Y}$. Let $\|\cdot\|_{Z_n(\widetilde{Y})}$ denote the ℓ_1-norm on $Z_n(\widetilde{Y})$ induced by a free $\mathbb{Z}G$-basis; by Remark 4.1.5 this is also filling norm on $Z_n(\widetilde{Y})$. Using the algebraic definition of FV_G^{n+1} (see Definition 4.2.3 and Remark 4.2.4) a representative of FV_G^{n+1} is given by

$$FV_G^{n+1}(k) = \max\left\{\|\gamma\|_{Z_n(\widetilde{Y})} \colon \gamma \in Z_n(\widetilde{Y}),\ \|\gamma\|_1 \leq k\right\}. \tag{5.9}$$

Since $C_{n+1}(\widetilde{X}) \subseteq C_{n+1}(\widetilde{Y})$ is a free factor, the ℓ_1-norm on $C_{n+1}(\widetilde{X})$ induced by the $(n+1)$-cells of $\widetilde{X}$ equals the restriction of $\|\cdot\|_1$ to $C_{n+1}(\widetilde{X})$. Let $\|\cdot\|_{Z_n(\widetilde{X})}$ denote the filling-norm on $Z_n(\widetilde{X})$ as a $\mathbb{Z}H$-module induced by the boundary map $\partial_{n+1} \colon C_{n+1}(\widetilde{X}) \to Z_n(\widetilde{X})$. Then (a representative of) FV_H^{n+1} is given by

$$FV_H^{n+1}(k) = \max\left\{\|\gamma\|_{Z_n(\widetilde{X})} \colon \gamma \in Z_n(\widetilde{X}),\ \|\gamma\|_1 \leq k\right\}. \tag{5.10}$$

Applying Lemma 4.1.10 to the short exact sequence (5.5), there exists a constant $C > 0$ and a homomorphism of $\mathbb{Z}H$-modules $\rho \colon Z_n(\widetilde{Y}) \to Z_n(\widetilde{X})$ such that $\rho \circ \imath$ is the identity on $Z_n\left(\widetilde{X}\right)$ and

$$\|\rho(\gamma)\|_{Z_n(\widetilde{X})} \leq C \cdot \|\gamma\|_{Z_n(\widetilde{Y})}, \tag{5.11}$$

for every $\gamma \in Z_n(\widetilde{Y})$.

Let $k \in \mathbb{N}$ and let $\gamma \in Z_n(\widetilde{X})$ such that $\|\gamma\|_1 \leq k$. Then (5.11) implies that

$$\|\gamma\|_{Z_n(\widetilde{X})} = \|\rho \circ \iota(\gamma)\|_{Z_n(\widetilde{X})} \leq C \cdot \|\iota(\gamma)\|_{Z_n(\widetilde{Y})} \leq C \cdot FV_G^{n+1}(k). \tag{5.12}$$

Since γ was arbitrary and $\|\gamma\|_1 = \|\imath(\gamma)\|_1$, this shows that $FV_H^{n+1}(k) \leq FV_G^{n+1}(k)$.

Remark 5.1.5 *The* cohomological *and* geometric *dimensions of a group G are defined respectively as*

$$cd(G) = \min\{ \ \mathit{length}(P_*) \mid P_* \textit{ is a projective resolution of } \mathbb{Z}G\text{–}\textit{modules for } \mathbb{Z} \ \}$$

$$gd(G) = \min\{ \ \mathit{dim}(X) \mid X \textit{ is a } K(G,1) \ \}.$$

The Eilenberg–Ganea Theorem states that the geometric and cohomological dimensions of a group are equal for all dimensions greater or equal to 3*. In the context of Theorem 5.1.4, if the initial group in question G is of dimension $n+1$, where $n \geq 2$, it cannot happen that $Z_m\left(\widetilde{M_f}\right)$ is stably finitely generated free as a $\mathbb{Z}G-$module for $m < n$ since*

$$0 \to Z_m\left(\widetilde{M_f}\right) \to C_m\left(\widetilde{M_f}\right) \to \cdots \to C_0\left(\widetilde{M_f}\right) \to \mathbb{Z} \to 0$$

would be a projective resolution of $\mathbb{Z}G-$modules for $\mathbb{Z}$ of length $m+1$, contradicting the fact that $cd(G) = n+1$. Thus our proof of Theorem 5.1.4 does not generalize to $m^{th}-$homological dehn functions for $m < n$. In general it is not true for such groups that $FV_H^{m+1} \preceq FV_G^{m+1}$; see Examples 5.1 and 5.2.

5.2 Non–examples

Example 5.1 (Non–hyperbolic Subgroups) *In [7] Brady constructed a group* G *admitting a finite* 3*–dimensional* $K(G,1)$ *with linear* FV_G^2 *containing a subgroup* $H \leq G$ *of type* F_2 *with* FV_H^2 *at least quadratic.*

Example 5.2 (Generalized Heisenberg Groups) *Other examples which contrast Theorem 5.1.4 include the generalized Heisenberg groups* $\mathcal{H}_{2n+1}$*, for which Young has computed the homological filling functions in [33,36]. For example,* $\mathcal{H}_5$ *admits a finite* 5*–dimensional* $K(\mathcal{H}_5,1)$ *and has quadratic* $FV_{\mathcal{H}_5}^2$*, while* $\mathcal{H}_3$ *can be embedded in* $\mathcal{H}_5$*, admits a* 3*–dimensional* $K(\mathcal{H}_3,1)$*, and has cubic* $FV_{\mathcal{H}_3}^2$*.*

Similar phenomena also occurs in higher dimension. For example, $\mathcal{H}_5$ *has quadratic* $FV_{\mathcal{H}_5}^3$*, and can be embedded in* $\mathcal{H}_7$ *which has* $FV_{\mathcal{H}_7}^3$ *polynomial of degree* $3/2$*.*

5.3 Applications

Recall that a group G is *hyperbolic* if it has a linear Dehn function. In [18] Gersten proved the following:

Theorem 5.3.1 *[18, Thm 4.6] Let* G *be a hyperbolic group of cohomological dimension* 2*. Then every finitely presented subgroup* H *of* G *is hyperbolic.*

Gersten's result does not hold in higher dimensions since the group G in Example 5.1 is of cohomological dimension 3 and the subgroup $H \leq G$ is finitely presented and not hyperbolic [7]. We can however obtain a result similar to Theorem 5.3.1 by considering homotopical filling invariants of higher dimensions.

Corollary 5.3.2 *Let G be a hyperbolic group of geometric dimension $n+1$. Let $H \leq G$ be of type F_{n+1}. Then δ^n_H is linear.*

Recall that the *geometric dimension* of a group G is the minimum dimension among $K(G,1)$'s. The Eilenberg–Ganea Theorem states that the cohomological and geometric dimensions of a group G are equal for dimensions greater or equal than 3. This justifies our use of geometric dimension as opposed to cohomological dimension in the above corollary. We also have the following homotopical version of Theorem 5.1.4 for sufficiently large dimensions.

Corollary 5.3.3 *Let $n \geq 3$. Let G be a group admitting a finite $(n+1)$–dimensional $K(G,1)$. Let $H \leq G$ be of type F_{n+1}. Then $\delta^n_H \preceq \delta^n_G$.*

Proof of Corollary 5.3.2 By the Eilenberg–Ganea Theorem, G admits an $(n+1)$–dimensional $K(G,1)$. Moreover, since G is hyperbolic, the $K(G,1)$ can be taken to be finite; see Theorem 7.2.1. Theorems 5.1.4 and 3.3.4 imply that FV^{n+1}_H is linear. It then follows from Theorem 3.3.1 that δ^n_H is linear.

Remark 5.3.4 *Corollary 5.3.2 does not apply to Brady's subgroup H in Example 5.1 as it is not of type F_3. It is an open question whether or not the subgroup H in Corollary 5.3.2 is in fact hyperbolic.*

Proof of Corollary 5.3.3 Apply Theorems 3.3.1 and 5.1.4.

Remark 5.3.5 *It is an open question whether or not the statement of Corollary 5.3.3 holds for dimensions 1 and 2. In general $\delta_G^1 \nsim FV_G^2$ and $\delta_G^2 \nsim FV_G^3$, so extra work is required. Examples of such groups are given in [1, 34].*

5.4 Some Remarks on δ_G and FV_G^2

We conclude our investigation of filling functions with a brief discussion about the relationship of δ_G and FV_G^2.

Definition 5.4.1 (Superadditive function) *Let $f\colon \mathbb{N} \to \mathbb{N}$. We define the* superadditive closure *of f to the the function*

$$s(f)(n) = \max_{1 \le r \le n} \{ \sum_{i=1}^{r} f(n_i) \mid \sum_{i=1}^{r} n_i = n,\ n_i \in \mathbb{N} \}.$$

We say f is superadditive *if $f \sim s(f)$.*

Remark 5.4.2 *It is easy to see from Definition 5.4.1 that $f(n) \le s(f)(n) \le n \cdot f(n)$.*

It is an open question whether or not δ_G is superadditive, Guba and Sapir have shown this is the case for the free product of two non–trivial finitely presented groups [20]. The following proposition is easy to prove and can be found in [16, Prop. 2.4].

Proposition 5.4.3 *Let G be a finitely presented group. Then $FV_G^2 \preceq s(\delta_G)$. Consequently, if δ_G is polynomial, then so is FV_G^2.*

Remark 5.4.4 *The converse of the second statement of Proposition 5.4.3 is not true. Abrams, Brady, Dani, and Young have constructed a group G with FV_G^2 polynomial and δ_G exponential [1].*

Remark 5.4.5 (Finiteness of FV_G^2) *Proposition 5.4.3 combined with Remark 5.4.2 implies that FV_G^2 is a finite–valued function for finitely presented groups.*

Baumslag–Solitar groups were introduced in [4] and have since become a very important class of groups within Geometric Group Theory for testing conjectures and new techniques.

Theorem 5.4.6 *[17, Thm A] The Baumslag–Solitar group $BS(m,n) = \langle\, a, b \mid ba^m b^{-1} = a^n \,\rangle$ satisfies $FV^2_{BS(m,n)}(k) \succeq 2^k$ provided $|m| \neq |n|$.*

The following result was know to Gersten, however he puts an unnecessary restriction that G be automatic, in which case $\delta_G(k) \preceq k^2$ [14]. This is likely because automatic groups were quite popular at the time and the term "Dehn function" had only just been introduced.

Theorem 5.4.7 *[17, Thm B] Let G be a group admitting a finite 2–dimensional $K(G,1)$ with polynomial Dehn function δ_G. Then G contains no Baumslag–Solitar groups of the form $BS(m,n)$, $|m| \neq |n|$.*

Proof Apply Theorems 5.1.4, 5.4.6, and Proposition 5.4.3.

Chapter 6

Appendix A: Examples of Filling Functions

To give examples of filling functions, it is more convenient (and common practice) to work with *van Kampen diagrams* as opposed to admissible maps. We will briefly recall the necessary definitions and provide some examples of low–dimensional filling functions of groups.

6.1 The Cayley Complex

Definition 6.1.1 (Cayley Graph) *Let G be a group and let S be a finite generating set for G. The* Cayley graph *of G for the generating set S is the directed graph $\Gamma_{G,S}$ with vertex set $V = G$ and edge set $E = \{ \ (g, gs_i) \mid g \in G, \ s_i \in S \cup S^{-1} \ \}$. Edges (g, gs_i) are directed such that the initial vertex is g and terminal vertex is gs_i.*

The Cayley graph is particularly useful as it allows one to study the Word problem for G graphically: by fixing a basepoint $x_0 \in \Gamma_{G,S}$, we obtain a 1–1 correspondence between words ω over $S \cup S^{-1}$ and edge paths γ_ω originating from x_0. Moreover, $\omega \equiv 1_G$ if and only if γ_ω is a loop.

If $\langle \ S \mid R \ \rangle$ is a (not necessarily finite) presentation for G, then the *presentation complex* for $\langle \ S \mid R \ \rangle$ is defined as the following 2–complex $\mathcal{P}$ consisting of a single 0-cell v, a 1–cell e_s for each $s \in S$, and a 2-cell f_r for each $r \in R$. The 1–cells are attached in the obvious way; identifying their endpoints to v to form a wedge sum of $|S|$ circles. By directing and labeling each e_s by s, one obtains a $1-1$ correspondence between words over $S \cup S^{-1}$ and edge paths in the 1–skeleton $\mathcal{P}^{(1)}$; for generators s, the letter s^{-1} corresponds to transversing e_s in the direction opposite to its assigned orientation, and the word $\omega = s_1 s_2 \ldots s_n$ corresponds the the concatenation of loops $e_{s_1}, e_{s_2}, \ldots, e_{s_n}$. For the 2–cells, one takes a regular $|r|$–gon f_r labeled (and appropriately directed) by r and identifies its boundary with the edge path corresponding to r in $\mathcal{P}^{(1)}$. By mapping the homotopy class of e_s to s, the Seifert–van Kampen Theorem ensures that $\pi_1(\mathcal{P}) \simeq G$; see [22, Prop 1.26, Corr 1.28] for details.

The universal cover $\widetilde{\mathcal{P}}$ is called the *Cayley* 2*–Complex* for the presentation $\langle\, S \mid R \,\rangle$ and its 1-skeleton $\widetilde{\mathcal{P}}^{(1)}$ can be identified with the Cayley graph $\Gamma_{G,S}$. Thus $\widetilde{\mathcal{P}}$ serves as a tool much like the Cayley graph $\Gamma_{G,S}$, but with the added benefit that $\widetilde{\mathcal{P}}$ is simply connected.

6.2 van Kampen Diagrams

Definition 6.2.1 (Diagram) *[28] Let $\langle\ S \mid R\ \rangle$ be a presentation for a group G. A* diagram *X over $\langle\ S \mid R\ \rangle$ is a finite combinatorial 2–complex with edges labelled and directed by $S \cup S^{-1}$ and boundaries of 2–cells labeled by $R \cup R^{-1}$*

Definition 6.2.2 (Length and Area) *If γ is an edge path in a diagram X over $\langle\ S \mid R\ \rangle$, then the* length *of γ is the number of edges in γ. If U is a subcomplex of X, the* area *of U is the total number of 2–cells in U.*

Definition 6.2.3 (van Kampen Diagram) *Let $G = \langle\ S \mid R\ \rangle$. A* van Kampen diagram *D for a word ω over $S \cup S^{-1}$ is an oriented contractible planar diagram D over $\langle\ S \mid R\ \rangle$ with a base point $p \in \partial(D)$ such that $\partial(D) = \omega$ when read from p.*

The fundamental result about van Kampen diagrams is the *van Kampen Lemma* which guarantees their existence. A rigorous and accesible proof of Theorem 6.2.4 can be found in [9].

Theorem 6.2.4 (van Kampen Lemma) *[9, Thm 4.2.2] Let $\mathcal{P} = \langle\ S \mid R\ \rangle$ be a presentation of a group G and let ω be a word over $S \cup S^{-1}$. Then*

(1) $\omega \equiv 1_G$ *if and only if there exists a van Kampen diagram* D *for* ω *over* $\mathcal{P}$.

(2) If $\omega \equiv 1_G$ *then*

$$\mathsf{Area}_{\mathcal{P}}(\omega) = \min\{\ \mathsf{Area}(D) \mid D \textit{ is a van Kampen diagram for } \omega \textit{ over } \mathcal{P}\ \}.$$

If D is a van Kampen diagram for ω, then there is a natural combinatorial map f from D to the presentation complex X which maps the 2–cells of D to the corresponding relator 2–cell of X. Since $\pi_1(D) = 0$, the Lifting Criterion provides a lifting $\tilde{f} \colon D \to \widetilde{X}$ of f to the Cayley complex $\widetilde{X}$. This gives meaning to the phrase "D fills ω" where D is a van Kampen diagram for ω, since $\tilde{f}$ maps $\partial(D)$ to the edge circuit corresponding to ω in $\widetilde{X}$. As a consequence of the van Kampen Lemma and the preceeding remarks, we can interpret the Dehn function geometrically as a filling function of the Cayley complex $\widetilde{X}$ which is similar in nature to Fill$^{\,0}_{\widetilde{\mathrm{M}}}$.

Proposition 6.2.5 (Lifting Criterion) *[Hatcher, Prop. 1.33] Let* $f \colon Y \to X$ *be a continuous map and let* $\widetilde{X}$ *be the universal cover of* X. *Then there exists a* lifting $\tilde{f} \colon Y \to \widetilde{X}$ *of* f *such that* $\rho \circ \tilde{f} = f$ *if and only if the induced homomorphism* $f_* \colon \pi_1(Y) \to \pi_1(X)$ *is trivial.*

6.3 Surface Groups

We introduce a combinatorial notion of curvature and derive a formula resembling the Gauss–Bonnet Theorem from Differential Geometry. This Theorem is used in Geometric Group Theory to analyze the geometry of van Kampen diagrams. In particular, we will use this combinatorial version of the Gauss–Bonnet Theorem to show that the fundamental group of the closed orientable surface of genus 2 is hyperbolic. This argument is based on ideas from small–cancellation theory and was shown to me by Eduardo Martinez–Pedroza. The argument can be easily modified to work for the fundamental group of an closed orientable surface of genus ≥ 2.

6.3.1 Combinatorial Curvature

Definition 6.3.1 (Link of a vertex) *Given a* 0*–cell* v *in a* 2*-complex* X*, the link of* v*, denoted* $link(v)$*, is the* 1*–complex formed by taking the midpoints of each* 1*-cell incident to* v *and connecting two such midpoints if the original* 1*-cells share a common* 2*–cell in* X*.*

Definition 6.3.2 *An angled* 2*-complex is a* 2*–complex* X *whose corners of* 2*-cells are labeled with non-negative real numbers called* angles*. If* c *is a corner of a* 2*–cell in* X*, we denote the angle assigned to* c *by* $\angle(c)$*.*

Our definition of curvature in angled 2–complexes is based on the defect of triangles.

Definition 6.3.3 (Curvature) *Let X be an angled 2-complex. The curvature of faces f and vertices v in X are defined as follows:*

- $\kappa(f) = \sum_{c \in Corners(f)} \angle(c) - (|\partial f| - 2)\pi$
- $\kappa(v) = 2\pi - \pi\chi(link(v)) - \sum_{c \in Corners(v)} \angle(c)$

where $|\partial f|$ is the number of 1-cells on the boundary of f, and $\chi(link(v))$ is the Euler characteristic of the link of v.

We prove the Combinatorial Gauss–Bonnet Theorem.

Theorem 6.3.4 (Gauss-Bonnet) *Given an angled 2-complex S,*

$$2\pi\chi(S) = \sum_{v \in S} \kappa(v) + \sum_{f \in S} \kappa(f).$$

Proof Let V, E, and F denote the number of vertices, edges, and faces in S respectively. By the definition of $\kappa(v)$ and $\kappa(f)$ we have

$$\begin{aligned}\sum_{v \in S} \kappa(v) + \sum_{f \in S} \kappa(f) = 2\pi V - \pi \sum_{v \in S} \chi\left(link(v)\right) - \sum_{v \in S} \left(\sum_{c \in \text{Corners}(v)} \angle(c) \right) \\ + \sum_{f \in S} \left(\sum_{c \in \text{Corners}(f)} \angle(c) \right) - \sum_{f \in S} \left(\left(|\partial f| - 2 \right) \pi \right).\end{aligned}$$

Since both $\sum_{v\in S}\left(\sum_{c\in \mathrm{Corners}(v)} \angle(c)\right)$ and $\sum_{f\in S}\left(\sum_{c\in \mathrm{Corners}(f)} \angle(c)\right)$ measure the sum of all angles in S, their difference is 0. This simplifies the right hand side of the above equality to

$$2\pi V - \pi \sum_{v\in S} \chi\left(link(v)\right) - \sum_{f\in S}\left((|\partial f| - 2)\pi\right).$$

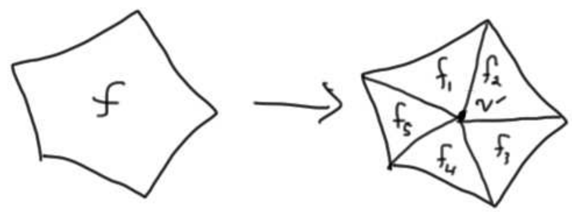
By appropriately subdividing S, we may assume all 2-cells in S are triangles (see Remark 6.3.5). Since $\chi(link(v)) = |\{\text{edges incident to v}\}| - |\{\text{faces incident to v}\}|$, the above expression becomes

$$2\pi V - \pi \sum_{v\in S}\left(|\{\text{edges incident to v}\}| - |\{\text{faces incident to v}\}|\right) - \pi \sum_{f\in S}(3-2)$$
$$= 2\pi V - \pi F - \pi \sum_{v\in S}\left(|\{\text{edges incident to v}\}| - |\{\text{faces incident to v}\}|\right).$$

However, in the above sum, each edge is counted twice and each triangle three times. Therefore we have

$$\sum_{v\in S}\kappa(v) + \sum_{f\in S}\kappa(f) = 2\pi V - \pi F - \pi(2E - 3F) = 2\pi\chi(S).$$

Remark 6.3.5 (Triangulation of S) *Triangulate every 2–cell in S in the manner illustrated below.*



Assign angles at each new (interior) vertex v' such that $\kappa(v') = 0$ and assign angles at each remaining vertex such that $\kappa(f) = \sum_{i=1}^{n} \kappa(f_i)$ for each n–gon f in S. This subdivision construction yields a new angled 2–complex S' such that

$$2\pi V - \pi \sum_{v \in S} \chi\left(link(v)\right) - \sum_{f \in S} \left(\left(|\partial f| - 2\right)\pi\right) = 2\pi V - \pi \sum_{v \in S'} \chi\left(link(v)\right) - \sum_{f \in S'} \left(\left(|\partial f| - 2\right)\pi\right).$$

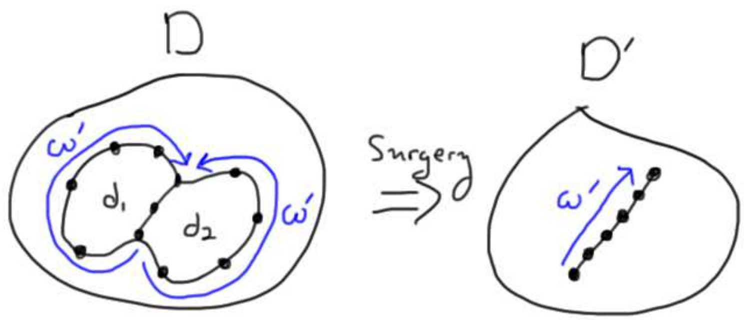
6.3.2 Hyperbolic Surface Groups

Definition 6.3.6 *Given a van Kampen diagram D, the intersection of two 2-cells in D is called a* piece*.*

Proposition 6.3.7 *Let G be the fundamental group of the genus 2 orientable closed surface with presentation $G = \langle\, a, b, c, d | [a, b][c, d] = 1 \,\rangle$. If D is a* minimal area *van Kampen diagram for a word $\omega = 1_G$, then D has no pieces of length greater than 1.*

Sketch Let D be a minimal area van Kampen diagram for some word $\omega \equiv 1_G$ and consider two intersecting two cells in D.

Observe that there are no subwords of length 2 or greater appearing in both $r = [a,b][c,d]$ and r^{-1} (or any cyclic permutation of the words). Therefore if two 2–cells, say d_1 and d_2, in D share a piece of length greater or equal to 2, then their boundaries (when read from their intersection) must be the same word. By removing d_1 and d_2 from D, we obtain a new van Kampen diagram D' for ω with $\mathsf{Area}(D') = \mathsf{Area}(D) - 2$. But this contradicts the fact that D was of minimal area, therefore all pieces in D have length 0 or 1.

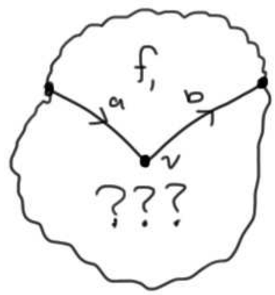


Definition 6.3.8 (Hyperbolic Group) *A finitely presented group G is hyperbolic if it has a linear Dehn function.*

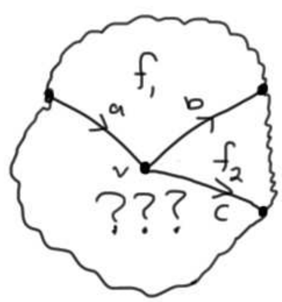
Theorem 6.3.9 *The fundamental group of the genus 2, orientable, closed, surface is hyperbolic.*

Sketch Let G be the fundamental group of the genus 2 orientable closed surface with presentation $G = \langle\, a, b, c, d | [a,b][c,d] = 1 \,\rangle$. We will prove that δ_G is a linear function by showing there exists a fixed constant $C > 0$ such that every word $\omega = 1_G$ satisfies $\mathsf{Area}(\omega) \leq C \cdot \text{length}(\omega)$.

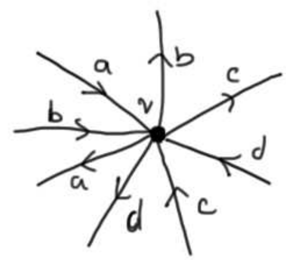
Let D be a minimal area van Kampen diagram for a word $\omega \equiv 1_G$. By Proposition 6.3.7, if v is an interior vertex of D, then we may assume v to have at least 3 incident edges. Without loss of generality, assume that one of the faces incident to v, say f_1, is as depicted below:



As previously stated, there must be another edge incident to v. Since all 2-cells have boundary $[a, b][c, d]$ or $([a, b][c, d])^{-1}$, and D is of minimal area, we can determine the labelling and orientation of this edge:



Continuing in this manner we determine that the local neighbourhood of any interior vertex of D is as shown below:



By Theorem 6.3.4 we have

$$2\pi = \sum_{v} \kappa(v) + \sum_{v_\infty} \kappa(v_\infty) + \sum_{f} \kappa(f) + \sum_{f_\infty} \kappa(f_\infty)$$

where v and f are interior vertices and faces, and v_∞ and f_∞ intersect the boundary of D. Note that we are free to assign angles to D as we see fit.

By assigning all interior angles to be $\pi/8$ and all boundary angles to be π, we have

- $\kappa(v) = 0$
- $\kappa(f) = -5\pi$
- $\kappa(v_\infty) = 2\pi - \pi(1) - \sum_{c \in Corners(v_\infty)} \pi \leq 0.$

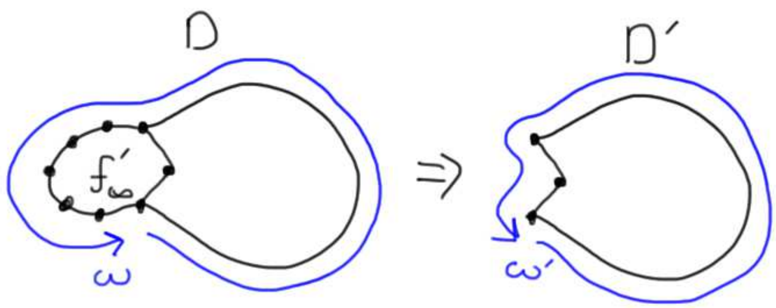
Therefore there exists a 2–cell f'_∞ intersecting the boundary of D such that $\kappa(f'_\infty) > 0$. Let c_∞ denote the corners of f'_∞ on the boundary of D and c be the corners of f'_∞ in the interior of D. Then

$$\kappa(f'_\infty) = \sum_{c_\infty} \pi + \sum_{c} \pi/8 - 6\pi > 0,$$

thus $8c_\infty + c > 48$ and $c_\infty + c = 8$. So it must be the case that $c_\infty \geq 6$.

By removing f'_∞ from D, we obtain a reduced van Kampen diagram D' for some new word $\omega' = 1_G$ such that:

- $\text{length}(\omega') + 4 \leq \text{length}(\omega)$, and
- $\mathsf{Area}(D') + 1 = \mathsf{Area}(D)$.



If we take K to be the area of D and iterate this process $K - 1$ times, we obtain a van Kampen diagram $\widehat{D}$ with area 1 for a word $\hat{\omega} = 1_G$ such that:

- $\text{length}(\hat{\omega}) + 4(K - 1) \leq \text{length}(\omega)$, and
- $Area(\widehat{D}) + K - 1 = Area(D)$.

Therefore $Area(D) = K \leq 4K + 4 \leq \ell(\omega)$. Since ω was arbitrary, δ_G is linear.

6.4 Baumslag–Solitar Groups

In this section we will study some low dimensional filling invariants of the Baumslag–Solitar group $BS(1,2)$ and its double $BS(1,2) \times BS(1,2)$.

6.4.1 The Baumslag–Solitar Group $BS(1,2)$

We will show that the filling functions δ_G, and FV_G^2 for the Baumslag–Solitar group $BS(1,2) = \langle\, a, b \mid aba^{-2}b^{-1} = 1 \,\rangle$ each grow faster than any polynomial function. This is done by constructing a family of words $\{\omega_k\}_{k\in\mathbb{N}}$ such that the length of each ω_k increases linearly with respect to k, but filling areas and filling norm increase exponentially. This is a standard argument and can be modified for Baumslag–Solitar groups $BS(m,n) = \langle\, a, b \mid ba^m b^{-1} = a^n \,\rangle$ where $|m| \neq |n|$; see [9, Excercise 7.2.11] and [17, Thm B]

For the remainder of this section it will be convenient to fix some notation. We will take X to be the Cayley complex associated to presentation $\langle\, a, b \mid aba^{-2}b^{-1} = 1 \,\rangle$ for $BS(1,2)$. Slightly abusing notation, we let ω denote the induced edge path in X for a word ω over $\{a, b, a^{-1}, b^{-1}\}$ and let $c_1(\omega)$ denote the induced 1–chain in $C_1(X)$. We let $\|\cdot\|_1$ denote the ℓ_1–norm on $C_i(X)$ with the natural basis, and let $\|\cdot\|_\partial$ denote the filling norm on $Z_1(X)$ induced by $\partial\colon C_2(X) \to Z_1(X)$. In particular, we will make use of the fact that X is a *contractible* 2–complex (see [24]).

We will need to recall two results, the proofs of which are applications of the Hurewicz Theorem.

Theorem 6.4.1 (Gersten's Lemma) *[18, Prop 3.2] Let X be the Cayley complex of a finitely presented group. Let ω be an edge circuit in X and let $c_1(\omega)$ be the induced 1–chain. Then*

$$\|c_1(\omega)\|_\partial = \min_{z \in Z_2(X)} \|c_2(D) + z\|_1$$

where $c_2(D)$ is the induced 2–chain of any *van Kampen diagram D for ω. In particular, if X is contractible, then $\|c_1(\omega)\|_\partial = \|c_2(D)\|_1$.*

Lemma 6.4.2 (Asphericity Lemma) *[9, Lemma 7.2.9] Let X be the Cayley complex of a finitely presented group. If D is an* embedded *van Kampen diagram for a word ω, meaning the map $D \to X$ is injective on 2–cells, and X is contractible, then $\mathsf{Area}(\omega) = \mathsf{Area}(D)$.*

Proposition 6.4.3 *Define $v_k = b^{-k}a^{2k+1}b^k a^{-(2k+1)2^k}$ for each $k \in \mathbb{Z}$. Then $v_k \equiv 1_{BS(1,2)}$ and*

$$FArea(v_k) = FArea^0(v_k) = \|v_k\|_\partial = (2k+1)(2^k - 1).$$

Proof Upon inspection of the Cayley complex X (see Figure 5.1) we verify that $v_k \equiv 1_{BS(1,2)}$ since the edge path corresponding to v_k is a simple closed curve. Moreover, there is a particularly obvious embedded disk diagram for v_k, call it Δ_k, namely the region bounded by v_k in the sheet of X that v_k lies in (see Figures 5.1 and 5.4).

Since X is contractible and 2–dimensional, and the map $\Delta_k \to X$ is injective on 2–cells, by Theorem 6.4.1 and Lemma 6.4.2 we have

$$\mathsf{Area}(v_k) = \|v_k\|_\partial = \mathsf{Area}(\Delta_k)$$

and the area of Δ_k is easily computed (see Figure 5.4) to be

$$\begin{aligned}\mathsf{Area}(\Delta_k) &= (2k+1) + 2(2k+1) + 4(2k+1) + \cdots + 2^{k-1}(2k+1)\\ &= (2k+1)\left(\sum_{i=1}^{k} 2^{i-1}\right)\\ &= (2k+1)(2^k - 1).\end{aligned}$$

Proposition 6.4.4 *Define* $\omega_k = b^{-k}a^{2k+1}b^k a b^{-k} a^{-(2k+1)} b^k a^{-1}$ *for each* $k \in \mathbb{Z}$. *Then* $\omega_k \equiv 1_{BS(1,2)}$, *length*$(\omega_k) = \|c_1(\omega_k)\|_1 = 8k$, *and*

$$\mathsf{Area}(v_k) = \|v_k\|_\partial = (2)(2k+1)(2^k - 1).$$

Proof The fact that length$(\omega_k) = 8k$ is obvious. As in Proposition 6.4.3, by looking at X one verifies that $\omega_k \equiv 1_{BS(1,2)}$ since ω_k is a simple closed curve. In particular, this implies that length$(\omega_k) = \|c_1(\omega_k)\|_1$. Moreover, there is an obvious embedded disk diagram for ω_k, call it D_k, which is comprised of two copies of Δ_k (see Figure 5.5). Applying Theorem 6.4.1 and Lemma 6.4.2 we have

$$\mathsf{Area}(w_k) = \|w_k\|_\partial = \mathsf{Area}(D_k)$$

where $\mathsf{Area}(D_k) = (2)(2k+1)(2^k - 1)$.

Corollary 6.4.5 *Let $G = BS(1,2)$. Then $2^k \preceq f(k)$ where f can be taken as δ_G, or FV_G^2.*

Proof Observe that the inequality $2^k \leq 2(2k+1)(2^k-1)$ is equivalent to

$$2^k + 4k + 2 \leq k2^{k+2} + 2^{k+1},$$

which holds for all positive integers k. Therefore

$$2^k \leq 2(2k-1)(2^k-1) \leq f(8k),$$

where the rightmost inequality follows from Proposition 6.4.4. Thus $2^k \preceq f(k)$.

6.4.2 The Double $BS(1,2) \times BS(1,2)$

We will now briefly discuss an interesting result of Robert Young concerning the filling functions δ_G^2 and FV_G^3 for the product $G = BS(1,2) \times BS(1,2)$.

Theorem 6.4.6 *[34, Corollary 6] Let $G = BS(1,2)$. Then $FV^3_{G\times G}(k^2) \succeq 2^k$ and $\delta^2_{G\times G}(k) \preceq k^2$.*

As before, take X to be the Cayley complex associated to $G = BS(1,2)$. Since X is 2–dimensonal and contractible, we have $\delta_G^2(k) \preceq k$. A result of Alonso et al. [2]

then states that $\delta^2_{G\times G}(k) \preceq k^2$. Young's argument for the inequality $FV^3_{G\times G}(k) \succeq 2^{\sqrt{k}}$ is based on constructing tori in $X \times X$ which require large amounts of volume to fill – this is outlined below. Note that the product $X \times X$ is the universal cover of a $K(G \times G, 1)$. We let $\|\cdot\|_1$ and $\|\cdot\|_\partial$ denote the natural ℓ_1 and filling norms on $C_i(X \times X)$ and $Z_i(X \times X)$.

Each word ω_k in $BS(1,2)$ can be interpreted as an injective admissible map $\alpha_k \colon S^1 \to X$ of volume $8k$. Using α_k, we can construct an injective admissible map $\alpha_k^2 \colon S^1 \times S^1 \to X \times X$ of volume $64k^2$. Since α_k^2 is injective, the induced 2–cycle of $X \times X$, call it T_k, has $\|T_k\|_1 = 64k^2$. We claim the filling norm of the torus T_k is exponential with respect to k.

Let β be a 3–chain such that $\partial(\beta) = T_k$ and $\|\beta\|_1 = \|T_k\|_\partial$. Let U be the 3–cells of $X \times X$ in the support of the 3–chain β and consider the projection maps $p_1,\ p_2 \colon X \times X \to X$. Making use of the Universal Coefficient Theorem, Young argues that the subcomplex $p_i(U) \subset X$ supports a 2–chain with boundary ω_k for either $i = 1$ or 2. Since the boundary map $\partial_X \colon C_2(X) \to C_1(X)$ is injective, $p_i(U)$ must contain at least $\|\omega_k\|_{\partial_X}$ 2–cells, each of which is the image of a 3–cell in U. Therefore $\|\beta\|_1 \geq \|\omega_k\|_{\partial_X}$, and hence $FV^3_{G\times G}(k^2) \succeq 2^k$.

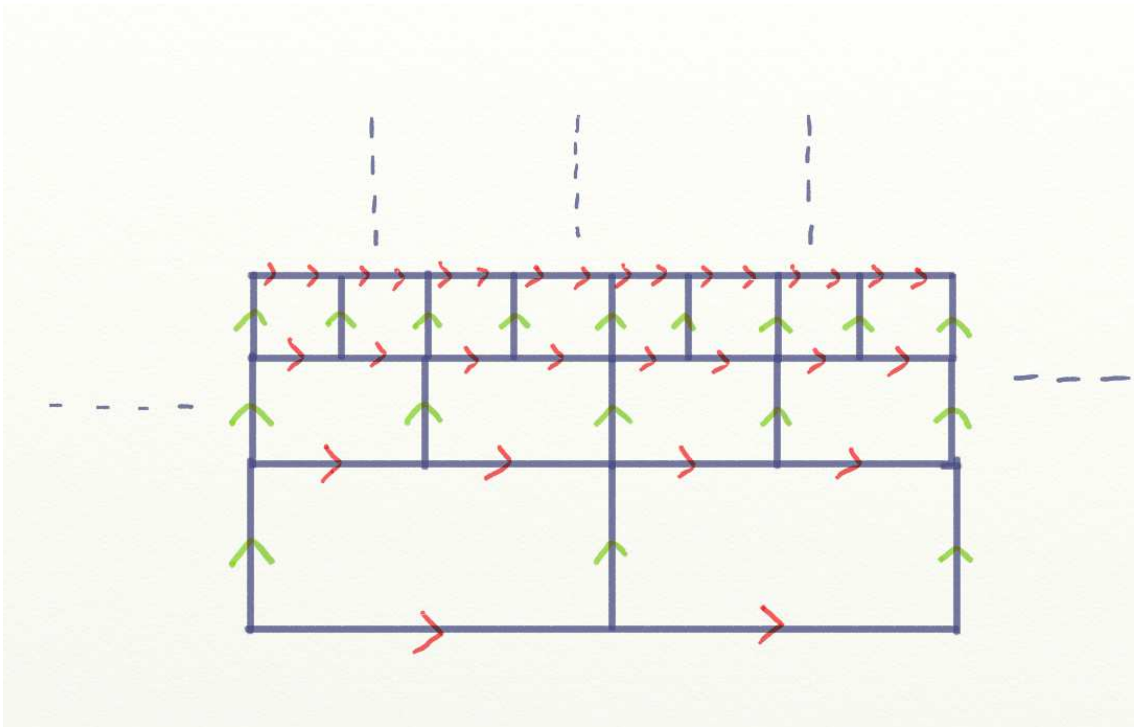


Figure 6.1: A Sheet of the Cayley Complex for the Baumslag–Solitar group $BS(1,2) = \langle\ a, b \mid ab = ba^2\ \rangle$.

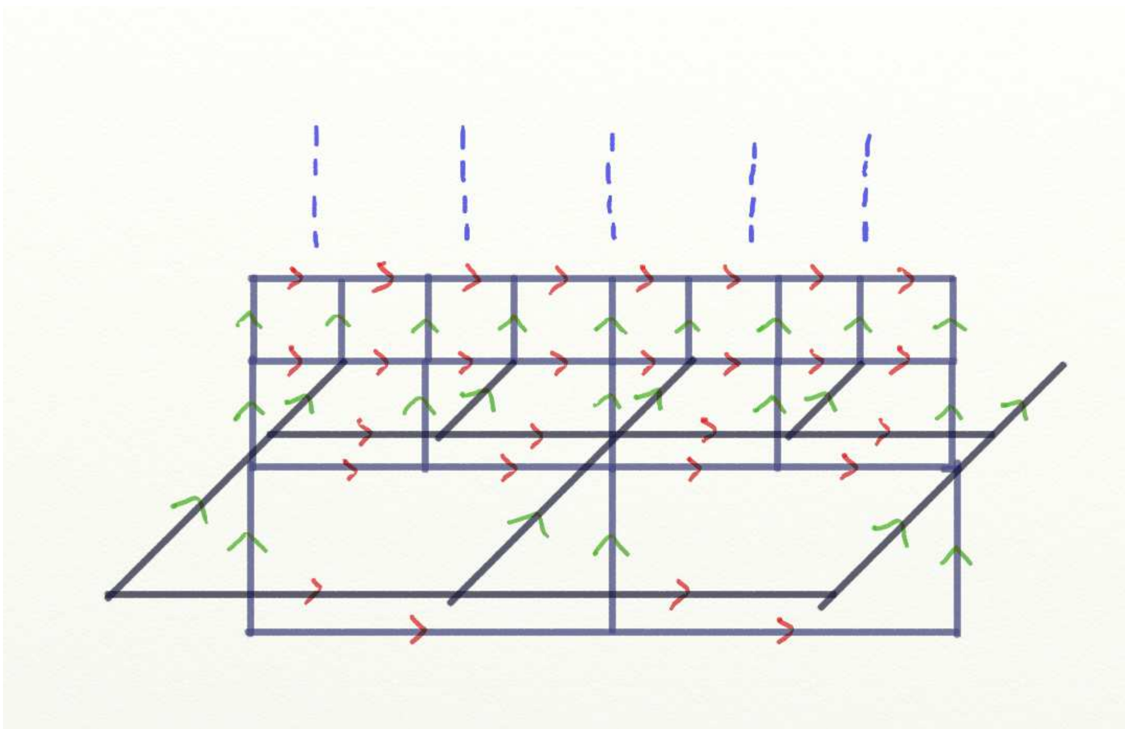


Figure 6.2: Two sheets of the Cayley Complex meeting.

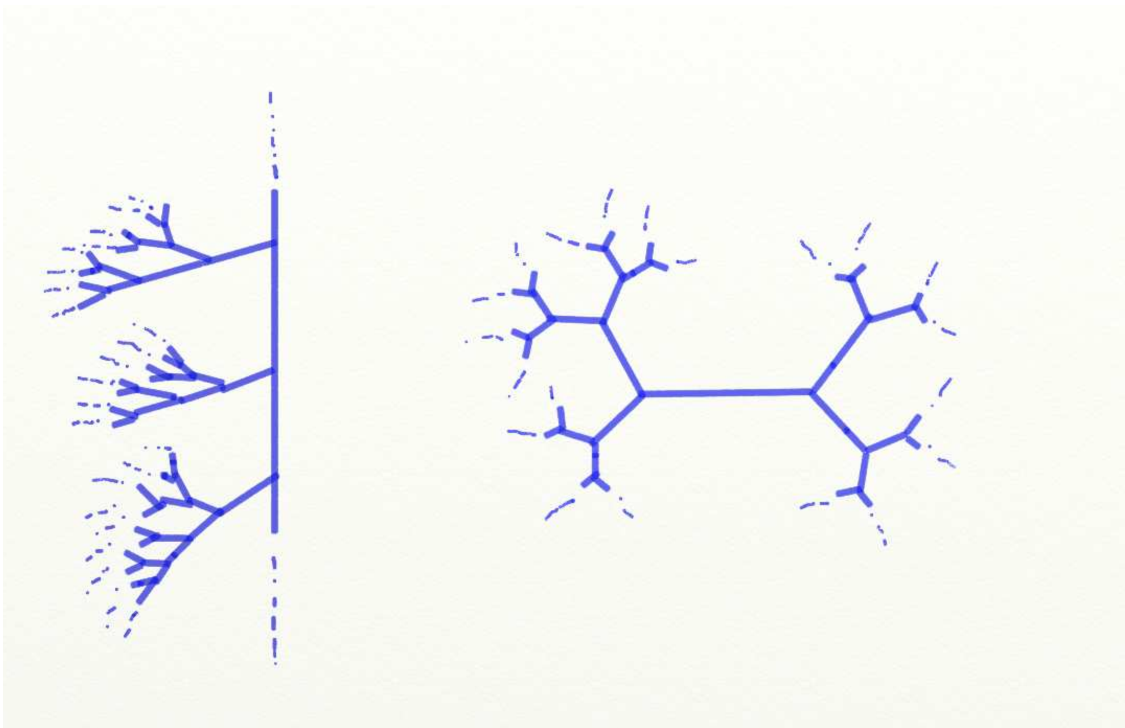


Figure 6.3: "Side View" of the Cayley Complex for $BS(1,2)$.

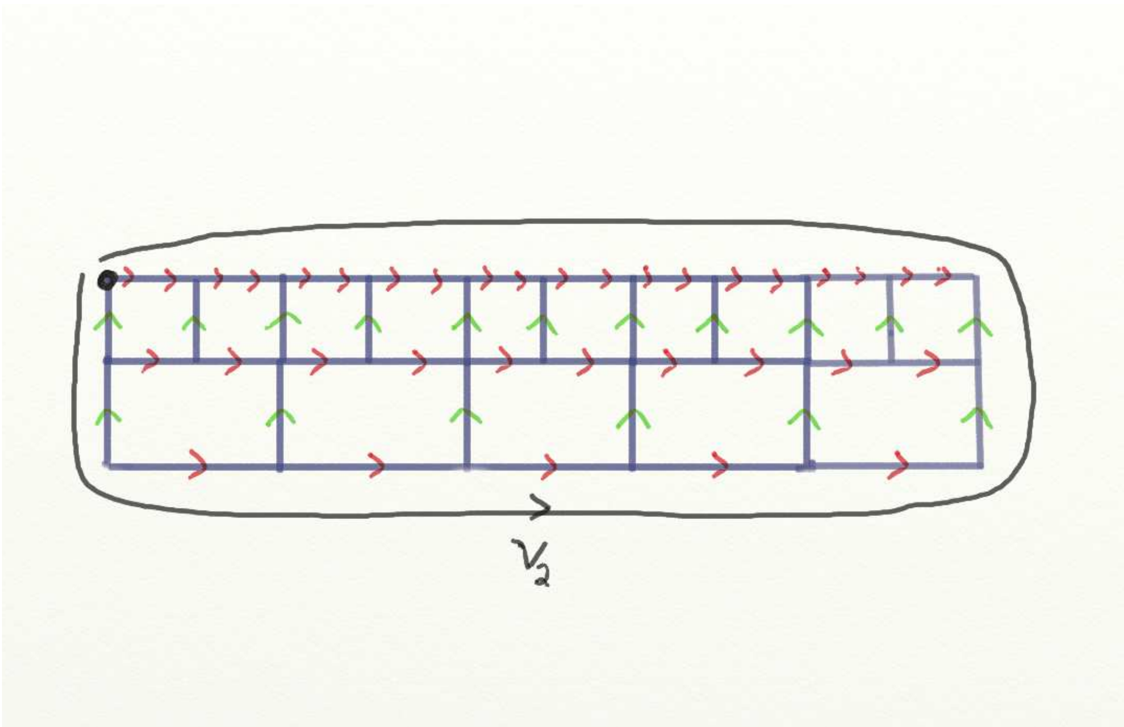


Figure 6.4: van Kampen Diagram for v_2.

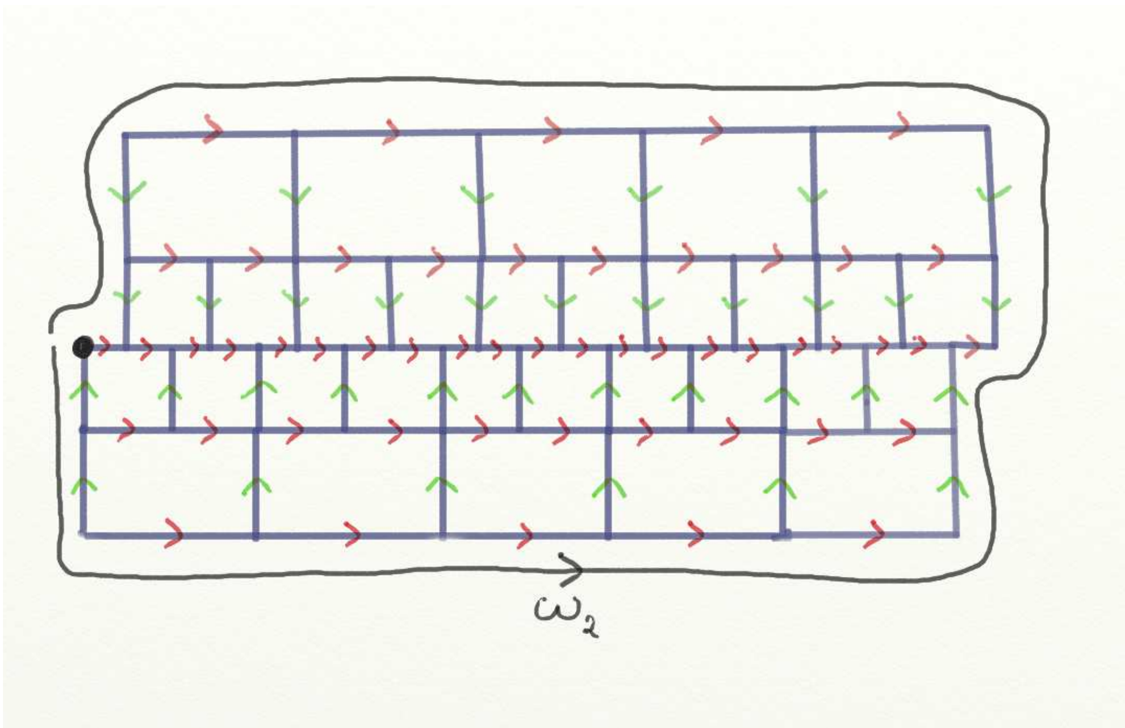


Figure 6.5: van Kampen Diagram for ω_2.

Chapter 7

Appendix B: The Eilenberg–Ganea Theorem

Definition 7.0.7 (Cohomological Dimension) *Let G be a group. The* cohomological dimension *of G is*

$$cd(G) = \min\{\ \mathit{length}(P_*) \mid P_* \text{ is a projective resolution of } \mathbb{Z}G\text{–modules for } \mathbb{Z}\ \}.$$

Definition 7.0.8 (Geometric Dimension) *Let G be a group. The* geometric dimension *of G is*

$$gd(G) = \min\{\ \mathit{dim}(X) \mid X \text{ is a } K(G,1)\ \}.$$

It is easy to see that $cd(G) \leq gd(G)$ since for any $K(G,1)$ on dimension n, the augmented cellular chain complex is a length n free resolution of $\mathbb{Z}G$–modules for $\mathbb{Z}$; see Theorem 8.2.25. What is not obvious, is that in most cases, the cohomological and geometric dimensions of a group are equal.

Theorem 7.0.9 (Eilenberg–Ganea Theorem) *Let G be a group with $cd(G) \geq 3$. Then $gd(G) = cd(G)$. If $cd(G) = 2$ then $gd(G) = 2$ or 3.*

We will prove a special case of Theorem 7.0.9 for hyperbolic groups, but in order to do so, we will need to make use of a number of powerful results.

7.1 The Hurewicz Theorem and Other Results

The n^{th} homotopy group of a space X , denoted $\pi_n(X)$, is defined similarly to the fundamental group of a space X, but uses homotopy classes of maps from n–spheres into the space X (see [15, Chapter 4.4] or [22]). Unlike the fundamental group, all higher homotopy groups of a topological space are Abelian.

Definition 7.1.1 (n–connected and n–acyclic spaces) *[15] A topological space is n–connected if it is path–connected and all higher homotopy groups $\pi_i(X)$, $1 \leq i \leq n$ are trivial. The space X is n–acyclic if it is path–connected and all higher homology groups $H_i(X)$, $1 \leq i \leq n$ are trivial.*

Theorem 7.1.2 (Hurewicz Theorem) *[15, Theorem 4.5.1] Let X be a simply connected cell complex and let $n \geq 2$. Then X is $(n-1)$–connected if and only if X is $(n-1)$–acyclic, in which case $\pi_n(X) \simeq H_n(X)$.*

The Hurewicz Theorem is particularily useful as it allows us to identify cellular n–cycles $\gamma \in Z_n(X)$ with the image of the fundamental class of an n–sphere $f : S^n \to X$.

Theorem 7.1.3 (Whitehead Theorem) *[15, Prop. 4.1.4] Let X be a path–connected space with $\pi_i(X) = 0$ for all $i > 0$. Then X is contractible.*

Theorem 7.1.4 (Rips Complex) *[10, Thm 3.21] Let G be a torsion–free hyperbolic group. Then G has a finite $K(G,1)$.*

7.2 Eilenberg–Ganea for Hyperbolic Groups

Theorem 7.2.1 *Let G be a hyperbolic group of $cd(G) = n$ where $n \geq 3$. Then G admits a* finite *n–dimensional $K(G,1)$.*

Sketch of Theorem 7.2.1 Take Y to be the finite $K(G,1)$ from Theorem 7.1.4. Since $cd(G) \leq gd(G)$, we can assume that $dim(Y) = m \geq n$. Let $\widetilde{Y}$ be the universal cover of Y and consider the following finite length resolution of finitely generated free

$\mathbb{Z}G$–modules:

$$0 \to C_m\left(\widetilde{Y}\right) \to C_{m-1}\left(\widetilde{Y}\right) \to \cdots \to C_0\left(\widetilde{Y}\right) \to \mathbb{Z} \to 0. \tag{7.1}$$

Restricting to the $(n-1)$–skeleton of Y, the following exact sequence of $\mathbb{Z}G$–modules arises:

$$0 \to Z_{n-1}\left(\widetilde{Y}\right) \to C_{n-1}\left(\widetilde{Y}\right) \to \cdots \to C_0\left(\widetilde{Y}\right) \to \mathbb{Z} \to 0. \tag{7.2}$$

Since $cd(G) = n$, there exists a projective resolution of $\mathbb{Z}G$–modules of length n for $\mathbb{Z}$. We can then use Schanuel's Lemma (Lemma 5.1.3) with this projective resolution and the sequence 7.2 to conclude that $Z_{n-1}\left(\widetilde{Y}\right)$ is a projective $\mathbb{Z}G$–module. Using Shanuel's Lemma again, this time on the sequences 7.1 and 7.2, we have that $Z_{n-1}\left(\widetilde{Y}\right)$ is stably finitely generated free. Let X' denote the space obtained by taking a wedge sum of a finite number of $(n-1)$–spheres such that $Z_{n-1}\left(\widetilde{X'}\right)$ is a finitely generated free $\mathbb{Z}G$–module. Since $n-1 \geq 2$, the Seifert–van Kampen Theorem guarantees that attaching $(n-1)$–spheres does not alter the fundamental group.

At this time, it is important to note the following properties of X':

- X' is finite and $(n-1)$–dimensional,
- $\pi_1(X') \simeq G$, since $n \geq 3$,
- $H_i\left(\widetilde{X'}\right) = 0$ for all $0 < i < n-1$.

Let $\{\gamma_1, \dots, \gamma_k\}$ be a free $\mathbb{Z}G$–basis for $Z_{n-1}\left(\widetilde{X}'\right)$. Using the Hurewicz Theorem, each γ_i can be realized as the image of the fundamental class of and $(n-1)$–sphere $S_i^{n-1} \to \widetilde{X}'$. By attaching k n–balls D_i^n to X' via the attaching maps given by the composition $\partial(D_i^n) = S_i^{n-1} \to \widetilde{X}' \to X'$, we obtain an n–dimensional cell complex X with $X^{(n-1)} = X'$ and fundamental group G. By definition, the boundary map $\partial_n \colon C_n\left(\widetilde{X}\right) \to Z_{n-1}\left(\widetilde{X}\right)$ is surjective. However, since $Z_{n-1}\left(\widetilde{X}\right)$ is free, the map ∂_n is also injective. Consequently, $\widetilde{X}$ is an n–dimensional, free, cocompact G–space with $H_i\left(\widetilde{X}\right) = 0$ for each $0 < i \leq n$. Furthermore, since $\widetilde{X}$ is simply connected, by the Hurewicz Theorem we have $\pi_i\left(\widetilde{X}\right) = 0$ for all $i > 0$. Whitehead's Theorem then implies $\widetilde{X}$ is contractible. As the G–action on $\widetilde{X}$ is free and cocompact, the quotient space $\widetilde{X}/G$ is a finite n–dimensional $K(G, 1)$.

Chapter 8

Appendix C: Technical Background

8.1 Algebraic Topology

8.1.1 Homotopy And The Fundamental Group

Definition 8.1.1 (Homotopy Equivalent Spaces) *[22] Let X and Y be topological spaces. Two continuous functions $f, g : X \to Y$ are* homotopic, *written $f \simeq g$, if there exists a continuous function $F : X \times [0, 1] \to Y$ such that*

$$F(x, 0) = f(x) \text{ and } F(x, 1) = g(x) \text{ for all } x \in X.$$

X and Y are homotopy equivalent *if there exists continuous functions $f : X \to Y$ and $g : Y \to X$ such that*

$$g \circ f \simeq id_X \text{ and } f \circ g \simeq id_Y.$$

Definition 8.1.2 (Path–connected) *[22] A* path *in a topological space X is a continuous function $f : [0,1] \to X$. We call such an f a* loop *if $f(0) = f(1)$. The space X is* path–connected *if for every two points $x, y \in X$ there exists a path $f\colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.*

Definition 8.1.3 (Path Homotopy) *[22, Prop 1.2] Two paths, f and g in X, are* path-homotopic *if there exists a continuous function $F : [0,1] \times [0,1] \to X$ such that:*

- *$F(0,t) = f(t)$ and $F(1,t) = g(t)$,*
- *the endpoints $F(s,0)$ and $F(s,1)$ remain fixed for all $s \in [0,1]$.*

If f and g are path-homotopic, we will write $f \sim g$. It is routine to verify that $\sim$ is an equivalence relation and we let $[f]$ denote the equivalence class of paths in X which are path-homotopic to f.

Definition 8.1.4 (Fundamental Group) *[22, Prop 1.3] The* fundamental group *of a topological space X with respect to a fixed basepoint $x_0 \in X$, is the group with set*

$$\pi_1(X, x_0) = \{\ [c] \mid c \text{ is a loop based at } x_0\ \}$$

and the following binary operation $*$ *which we call* concatenation*:*

If $[c]$ *and* $[c']$ *are the equivalence classes of loops* c *and* c' *based at* x_0 *in* X*, the concatenation of* $[c]$ *and* $[c']$ *is defined as the equivalence class* $[c' * c]$ *of the loop* $c' * c : [0,1] \to X$ *defined by:*

$$(c' * c)(t) = \begin{cases} c(2t), & \text{for } 0 \leq t \leq \frac{1}{2} \\ c'(1-2t), & \text{for } \frac{1}{2} < t \leq 1 \end{cases}$$

Proposition 8.1.5 *[22, Prop.* 1.5, 1.18*] If* X *and* Y *are homotopy equivalent path-connected spaces then* $\pi_1(X, x_0) \simeq \pi_1(Y, y_0)$*, regardless of choice of basepoints. In particular,* $\pi_1(X, x_0) \simeq \pi_1(X, x_0')$ *for any* $x_0, x_0' \in X$*, provided* X *is path-connected.*

Remark 8.1.6 (Notational Convention) *From now on we will simply write* $\pi_1(X)$ *for the fundamental group of a space* X *as all spaces will be assumed to be path-connected.*

8.1.2 Covering Spaces and Deck Transformations

Definition 8.1.7 (Universal Cover) *[22] Let* X *be a (locally path–connected and semi–locally simply path–connected) topological space. A* covering space *of* X *is a space* $\widehat{X}$ *together with a continuous surjective function* $\rho : \widehat{X} \to X$ *satisfying the following condition:*

- *for all points $x \in X$, there exists an open neighbourhood U_x of x such that $\rho^{-1}(U_x)$ is a union of disjoint open sets in $\widehat{X}$, each of which is mapped homeomorphically to U_x by ρ.*

The universal cover *of X, denoted $\widetilde{X}$, is a covering space $\rho\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is* simply connected *- meaning all loops in $\widetilde{X}$ are path-homotopic to the constant loop. The existence and uniqueness (up to homeomorphism) of universal covers can be found in [22, pg. 64].*

Remark 8.1.8 *Since we do not wish to get caught up in technicalities, all spaces which we consider are assumed to be locally path–connected and semi–locally simply path–connected; see [22] for definitions.*

Definition 8.1.9 (Group of Deck Transformations) *[22] Let $\rho : \widetilde{X} \to X$ be the universal cover of X and consider the set G of homeomorphisms $f : \widetilde{X} \to \widetilde{X}$ such that the following diagram commutes:*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "f"] \arrow[d, "\rho"] & \widetilde{X} \arrow[dl, "\rho"] \\ X & \end{tikzcd}$$

Then G forms a group whose operation is composition. We call G the group of deck transformations *of $\widetilde{X}$.*

Theorem 8.1.10 (Action of $\pi_1(X)$ on $\widetilde{X}$) *[22, Prop 1.39] Let $\rho : \widetilde{X} \to X$ be the universal cover of X and let G be the group of deck transformations of $\widetilde{X}$. Then*

i) G acts freely on $\widetilde{X}$,

ii) $G \simeq \pi_1(X)$.

iii) Any choice of preimage $\rho^{-1}(X^{(i)} - X^{(i-1)})$ in $\widetilde{X}$ forms a set of representative for the G–orbits of i–cells in $\widetilde{X}$.

Definition 8.1.11 ($K(G,1)$) *[22] A space X is* contractible *if it is homotopy equivalent to a point. A $K(G,1)$, is a cell complex X with contractible universal cover $\widetilde{X}$ and fundamental group isomorphic to G.*

Theorem 8.1.12 *[15, Prop. 7.1.5] Every group G admits a $K(G,1)$.*

8.1.3 Mapping Cylinders

One of the most useful traits of the fundamental group is its *functorial properties*. In particular, any continuous map $f : X \to Y$ induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.

Theorem 8.1.13 *[15, 22, Thm 1.4.3, Prop 1B.9] Let $H = \pi_1(X)$ and $G = \pi_1(Y)$ where Y is a $K(G,1)$. Then any homomorphism $\varphi : H \to G$ occurs as the induced* *homomorphism f_* for some continuous map $f : X \to Y$. Moreover, if X and Y are cell complexes, then f can be taken as a cellular map.*

Theorem 8.1.13 becomes quite useful when used in combination with *mapping cylinders.*

Definition 8.1.14 (Mapping Cylinder) *[31, Exercise 1.34] Given a continuous map $f : X \to Y$, the mapping cylinder of f is the quotient space*

$$M_f = ((X \times [0,1]) \sqcup Y)/\sim$$

where $(0,x) \sim f(x)$. In particular, M_f is homotopy equivalent to Y.

Remark 8.1.15 *In Definition 8.1.14 M_f can be thought as the space obtained by gluing the bottom of the "cylinder" $X \times [0,1]$ to the image $f(X) \subset Y$.*

Remark 8.1.16 *Let Y be a $K(G,1)$ and let $\pi_1(X) \simeq H$, where X is a cell complex and H is a subgroup of G. By Theorem 8.1.13 there exists a cellular map $f : X \to Y$ such that the induced homomorphism on fundamental groups is the inclusion $f_* : H \hookrightarrow Y$. In general, the map f is not necessarily an inclusion, however, by taking the mapping cylinder of f we may identify X with a subcomplex of M_f. This turns out to be quite a useful trick and will be used in the proof of Theorem 5.1.4.*

8.2 Homological Algebra

8.2.1 Free and Projective Modules

We will work under the assumption that all rings R are associative and have distinct multiplicative and additive identities, 1_R and 0_R. We now recall some basic facts about free and projective modules.

Definition 8.2.1 (Generating Set) *Let M be an R–module. A subset $S \subset M$ is a* generating set *for M if for every $m \in M$, there exists a finite linear combination $\sum r_i.s_i$ with $r_i \in R,\ s_i \in S$ such that $\sum r_i.s_i = m$. If M has a finite generating set, then M is* finitely generated*.*

Definition 8.2.2 (Free Module) *An R–module F is* free *if $F \simeq \bigoplus_E R$. That is to say, there exists a generating set $E \subset F$ such that if*

$$r_1.e_1 + \ldots r_n.e_n = 0_F,$$

where n is finite, $r_i \in R$, and each e_i is a distinct element of E, then

$$r_1 = r_2 = \cdots = r_n = 0_F.$$

Such a generating set E is called a free R–basis*.*

Remark 8.2.3 (Uniqueness of Coefficients) *It follows from Definition 8.2.2 that if F is a free R–module with free R–basis E, then every x in F can be expressed as a* unique *finite linear combination of the elements of E.*

Lemma 8.2.4 (Universal Property of Free Modules) *Let F and M be R-modules with F free with basis X. Then for any function $f : X \to M$, there exists a unique homomorphism $\varphi : F \to M$ extending f.*

Corollary 8.2.5 (Finitely Generated Module) *A R–module M is finitely generated if and only if there exists a surjective homomorphism $\pi\colon F \to M$ where F is free with a finite R–basis.*

Definition 8.2.6 (Projective Module) *An R-module M is* projective *if there exists an R-module N such that $M \oplus N$ is free.*

Lemma 8.2.7 (Splitting Lemma) *Let $0 \to M \to N \xrightarrow{\pi} P \to 0$ be a short exact sequence of R-modules which* splits*; meaning there exists an R–module homomorphism $\rho : P \to N$ such that $\pi \circ \rho = id_P$. Then $N \simeq M \oplus P$.*

Proposition 8.2.8 (Projective Modules) *The following are equivalent definitions for an R-module P to be projective:*

i) There exists M such that $P \oplus M$ is free.

ii) Given a homomorphism $f : P \to M$ and a surjective homomorphism $g : N \twoheadrightarrow M$, there exists a homomorphism $h : P \to N$ such that $f = g \circ h$.

iii) Every short exact sequence of R-modules $0 \to M \to N \to P \to 0$ splits.

8.2.2 The Integral Group Ring $\mathbb{Z}G$

Definition 8.2.9 (Group Ring) *[11] Let G be a group and let $\mathbb{Z}G$ be the free Z–module with basis G. The multiplication $\cdot : G \times G \to G$ extends uniquely to a $\mathbb{Z}$–bilinear product $\mathbb{Z}G \times \mathbb{Z}G \to \mathbb{Z}G$ giving $\mathbb{Z}G$ a ring structure. We call $\mathbb{Z}G$ the* integral group ring.

Proposition 8.2.10 *[11, Chap 1, Prop. 3.1] Let G act on a set X and let $\mathbb{Z}(X)$ denote the free abelian group with basis X. Then the G–action on X extends linearly to a $\mathbb{Z}G$–action on $\mathbb{Z}(X)$, hence $\mathbb{Z}(X)$ is a $\mathbb{Z}G$–module. Moreover, if G acts freely on X and E is a collection of representatives for the G–orbits in X, then $\mathbb{Z}(X)$ is a free $\mathbb{Z}G$ module with basis E.*

Remark 8.2.11 ($\mathbb{Z}$ is a $\mathbb{Z}G$–module) *Observe that $\mathbb{Z}$ is a $\mathbb{Z}G$–module since it arises from the trivial action of a group G on a point $\{x\}$.*

Proposition 8.2.12 *[11] Let G be a group and let $H \leq G$. Let G/H denote the collection of distinct left cosets of H in G. Then $\mathbb{Z}G \simeq \bigoplus_{G/H} \mathbb{Z}H$ as a $\mathbb{Z}H$–module.*

Corollary 8.2.13 *Let G be a group and let $H \leq G$. Then any free $\mathbb{Z}G$–module F can also be viewed as a free $\mathbb{Z}H$–module.*

8.2.3 The Long Exact Homology Sequence

Definition 8.2.14 (Homology Modules) *Let C_* be a (decreasing) chain complex of R–modules. Then n^{th} homology module of C_* is the quotient module*

$$H_n(C_*) = \ker \partial_n / \, im \, \partial_{n+1}.$$

If $H_n(C_*) = 0$ for all n, then C_* is said to be acyclic.

Theorem 8.2.15 (Long Exact Homology Sequence) *Given a short exact sequence of chain complexes*

$$0 \to A_* \to B_* \to C_* \to 0$$

there exists an exact sequence of homology modules

$$\cdots \to H_{n+1}(A_*) \to H_{n+1}(B_*) \to H_{n+1}(C_*) \to H_n(A_*) \to H_n(B_*) \to H_n(C_*) \to \dots.$$

Corollary 8.2.16 *[31, Exercise 5.15] Let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes. If two of A_*, B_*, C_* are acylic, so is the third.*

8.2.4 The Fundamental Lemma of Homological Algebra

Definition 8.2.17 (Free and Projective Resolutions) *[11] Let M be an R–module. A resolution of R–modules for M is an exact sequences of R–modules*

$$0 \to P_n \to \cdots \to P_0 \to M \to 0.$$

If each P_i is projective (free), then the resolution is a projective (free) *resolution. The* length *of such a resolution is n.*

Remark 8.2.18 *We also refer to infinite exact sequences of the form*

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_0 \to M \to 0,$$

where each P_i is projective, as projective resolutions of infinite length.

Definition 8.2.19 (Chain Homotopy) *Let (C_*, ∂_*) and (C'_*, ∂'_*) be chain complexes and $f_*, g_* \colon (C_*, \partial_*) \to (C'_*, \partial'_*)$ be chain maps. Then f_* and g_* are* chain homotopic, *written $f \simeq g$, if there is a sequence of homomorphisms $h_* = \{h_n \colon C_n \to C'_{n+1}\}$ such that for every n*

$$\partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n = f_n - g_n.$$

Such a sequence h_* is called a chain homotopy.

Definition 8.2.20 (Chain Homotopy Equivalent) *Two chain complexes (C_*, ∂_*) and (C'_*, ∂'_*) are* chain homotopy equivalent *if there exists chain maps $f_* \colon (C_*, \partial_*) \to (C'_*, \partial'_*)$ and $g_* \colon (C'_*, \partial'_*) \to (C_*, \partial_*)$ such that*

$$g_* \circ f_* \simeq id_{C_*} \text{ and } f_* \circ g_* \simeq id_{C'_*}.$$

Lemma 8.2.21 (Fundamental Lemma of Homological Algebra) *[11, Chap 1, Thm 7.5] Any two projective resolutions of an R–module M are chain homotopy equivalent.*

8.2.5 Cellular Homology

Definition 8.2.22 ((Augmented) Cellular Chain Complex) *[15] Let X be a cell complex and let A be a subcomplex of X. The group of cellular n–chains for the pair (X, A), denoted $C_n(X, A)$, is the free Abelian group with basis the n–cells of X not contained in A. For each $n \geq 1$, there exists a well defined homomorphism $\partial_n \colon C_n(X, A) \to C_{n-1}(X, A)$ and a homomorphism $\varepsilon \colon C_(X, A) \to \mathbb{Z}$ such that*

$$\cdots \to C_n(X, A) \xrightarrow{\partial_n} C_{n-1}(X, A) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} C_0(X, A) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is a chain complex of Abelian groups.

We refer to the above chain complex the (augmented) cellular chain complex of the pair (X, A) *and denote it by $C_*(X, A)$. We write $H_n(X, A)$ for the n^{th} homology group $H_n(C_*(X, A))$.*

Remark 8.2.23 (Notational Convention) *In Definition 8.2.22, if $A = \emptyset$ we will simply write $C_*(X)$ for the cellular chain complex of the pair $(X, \emptyset)$.*

Theorem 8.2.24 (Long Exact Homology Sequence of a Pair) *Let X be a cell complex and let $A \subset X$ be a subcomplex. Then there exists a short exact sequence of chain complexes*

$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$$

and a long exact sequence of homology groups

$$\cdots \to H_{n+1}(A) \to H_{n+1}(X) \to H_{n+1}(X,A) \to H_n(A) \to H_n(X) \to H_n(X,A) \to \ldots.$$

Theorem 8.2.25 (Free $\mathbb{Z}G$–resolution from $K(G,1)$) *Let X be a $K(G,1)$ for a group G. Then the free G–action on $\widetilde{X}$ endows the cellular chain complex $C_*\left(\widetilde{X}\right)$ with the structure of a free resolution of $\mathbb{Z}G$–modules for the trivial $\mathbb{Z}G$–module $\mathbb{Z}$. Moreover, if $X^{(n)}$ is finite, then each $C_i\left(\widetilde{X}\right)$ is finitely generated for $i \leq n$.*

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